AD-A103 857 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
ON BOUNDARY LAYER PROBLEMS IN THE THEORY OF ORDINARY DIFFERENTI--ETC(U)
JUL 81 W R WASOW DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2244 NL

MRC Technical Summary Report #2244

ON BOUNDARY LAYER PROBLEMS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Wolfgang R. Wasow

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

July 1981

(Received April 6, 1981)

This report is essentially a reprinting of Wolfgang Wasow's Ph.D. dissertation written in 1941.

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

Accession For	
NTIS GRA&I	☑
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A	

ON BOUNDARY LAYER PROBLEMS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Wolfgang R. Wasow

Technical Summary Report #2244

July 1981

SUMMARY

This research concerns linear ordinary differential equations depending in such a way on a parameter ρ that the "limit" differential equation obtained by letting ρ tend to ∞ in the differential equation is of lower order than the original one.

Adopting a term customary in physics we used the name <u>boundary layer problem</u> for the question: What happens to the solution of a boundary value problem of such a differential equation, if the parameter tends to ∞ in this solution?

We gave a general answer to this question for the differential equation $\frac{1}{\rho} N(y) + M(y) = 0$, where $N(y)$ and $M(y)$ are linear differential expressions of order n and m, respectively $(n > m)$, and for non-homogeneous boundary conditions which consist in prescribing the values of derivatives (but not of linear combinations of such derivatives) at the end-points. The question whether the solution of such a boundary value problem converges to a solution of the limiting differential equations, as $\rho \to \infty$, and what boundary conditions are satisfied by the limit function could be

The preparation of the report is sponsored by the United States Army under Contract No. DAAG29-80-C-0041

decided by an easily applicable rule. This rule showed, among other things that the solution converges only, if the prescribed n boundary conditions are not too unevenly distributed between the two endpoints.

If the order m of the limiting differential equation is only one less than the order n of the original differential equation, then the above mentioned rule could be extended to more general types of boundary conditions and also to non-homogeneous differential equations.

Since the most important boundary layer problems in the applications are concerned with systems of differential equations, we gave a simple example for the mathematical treatment of a boundary layer problem for a linear system of two simultaneous differential equations.

The validity of the general rule proved in this research was seen to be restricted by the assumption that the coefficient of the term of highest order of differentiation in $M(y)$ has no zeros in the interval of integration. In a special example we showed that interesting results can be obtained, if this assumption is dropped.

The theory of the asymptotic expansion of the solutions of linear differential equations involving a parameter, developed by G. D. Birkhoff, Noaillon, Tamarkin, Trjitzinsky and others proved an important and powerful tool in these investigations.

AMS (MOS) Subject Classification: 34E15

Key Words: Ordinary differential equations, Parameter, Boundary conditions, Asymptotic approximations, Boundary layers.

Work Unit Number 1 - Applied Analysis

COMMENT

In May 1980, the Mathematics Research Center organized a successful Advanced Seminar on Singular Perturbations and Asymptotics in honor of the retirement of a colleague, Wolfgang R. Wasow. His fundamental research is responsible for many other rapid developments in this field since 1940, and continues to play a vital role in modern theory and current applications. Wasow's Ph.D. dissertation (N.Y.U., 1941), a small part of which exists in print (On the asymptotic solution of boundary value problems for ordinary differential equations containing a parameter, J. of Mathematics and Physics 32 (1944), 173-183), represents the starting point of this important flourish of modern applicable research.

Following suggestions of several participants MRC is printing his 1941 thesis in its entirety as a TSR in order to make this valuable work more widely available. Readers will note that the name "singular perturbations" (which was only coined several years later by K. O. Friedrichs or W. Wasow or possibly jointly, but neither is now able to recall the details) does not appear anywhere explicitly!

TABLE OF CONTENTS

ON BOUNDARY LAYER PROBLEMS IN THE THEORY OF
ORDINARY DIFFERENTIAL EQUATIONS

Wolfgang R. Wasow

Introduction

Many problems in applied mathematics lead to questions of the following type:

> Given is a differential equation involving a parameter ρ. This parameter occurs in such a way that the "limiting" differential equation, i.e. the differential equation obtained by letting ρ tend to infinity _in the differential equation_, is of lower order than the original one. What happens then to the solution of a boundary value problem of the original differential equation, if ρ tends to infinity _in that solution_?

It is by no means obvious - and not even always true, as we shall see - that the solution of such a boundary value problem tends to a solution of the limiting differential equation, as ρ tends to infinity. But even when this is the case the question arises what are the boundary conditions satisfied by the limiting function. As a solution of a differential equation of _lower_ order than the original one it cannot, in general, be expected to satisfy all the boundary contions prescribed in the original problem.

In those cases in which the solution of the original problem converges - as ρ tends to infinity - to a solution of the limiting differential equation which does no longer satisfy all the originally prescribed boundary conditions, the solution f the original problem shows a peculiar behavior for very large values of the parameter ρ. Some of the derivatives of the solution will assume very large values in a narrow region near the boundary. As ρ tends to infinity, these derivatives will tend to infinity at a certain

The preparation of the report is sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

part of the boundary. In the most important applications of phenomena of this type the first derivative of the solution - and, of course, all the higher derivatives - diverge at parts of the boundary, as ρ tends to infinity.

In the physical interpretations this means the occurrence of "Boundary layers" in which the quantity to be investigated increases or decreases very rapidly with the distance from the boundary, when some physical constant is large. We shall use the name Boundary Layer Problems, in a more general sense, for all related mathematical problems.

The most famous problem of this type is the relationship between the theories of viscous and ideal liquids. An interesting boundary layer problem for a system of two non-linear ordinary differential equations has been investigated recently by K. Friedrichs and J. J. Stoker in a paper on the buckling of elastic plates, [9].

The majority of the applications lead to non-linear partial differential equations which are so complicated that a complete mathematical treatment has not yet been attempted. But even the boundary layer problem for linear ordinary differential equations, a problem interesting from the mathematical as well as from the physical point of view, has as yet been hardly investigated. The only paper known to the author of this investigation, on this problem for ordinary differential equations is the article [7] by Erich Rothe, in which the problem is solved for a very special linear differential equation of the second order with constant coefficients.

In Chapter I of the present paper we discuss the boundary layer problem for linear homogeneous differential equations depending linearly on the parameter, and for non-homogeneous boundary conditions. The result of this part can be expressed by a simple and easily applicable rule which determines immediately, for a given problem of this type, whether the solution converges and what boundary conditions are satisfied by the limiting function.

In Chapter II we investigate more thoroughly the case where the order of the limiting differential equation is lower by one than the order of the original differential equation. In this case the statement of the general rule of the Main Theorem in Chapter I can be formulated so as to include more general boundary conditions than those assumed in Chapter

I. The problem is then solved - at leact for a drop of one in the order of the differential equation - for the non-hor jeneous equation. Finally, the usual method of treating boundary layer problems, consisting of a transformation of the independent variable and an appropriate modification of the boundary conditions, is shown to be justified in this case. The method is sometimes used in more complicated problems without mathematical justification.

In Chapter III some special examples of other types of boundary layer problems are discussed.

The methods used in this paper are based on the theory of the asymptotic solution of ordinary differential equations involving a parameter. This theory has been developed by G. D. Birkhoff [1], Noaillon [2], Tamarkin [3], [4], Trjitzinsky [6], and others. In the Appendix we give a short outline of the results of this theory as far as they are used in this investigation.

I am deeply indebted to the Professors R. Courant and K. O. Friedrichs whose help and encouragement played a major part in the preparation of this thesis. The original suggestion for this investigation came from Prof. Friedrichs, and his active interest in the progress of the work has been of the utmost value.

Chapter I

THE BOUNDARY LAYER PROBLEM FOR THE DIFFERENTIAL EQUATION $\frac{1}{\rho} N(y) + M(y) = 0$

§1. Statement of the Problem

1. We consider the ordinary linear differential equation

$$L(y,\rho) = 0 \quad , \tag{101}$$

where the linear differential expression $L(y,\rho)$ is of the form

$$L(y,\rho) \equiv \frac{1}{\rho} N(y) + M(y) \tag{102}$$

with

$$N(y) \equiv \sum_{\nu=0}^{n} a_\nu(x)\, y^{(n-\nu)}(x) \tag{103}$$

$$M(y) \equiv \sum_{\mu=0}^{m} b_\mu(x)\, y^{(m-\mu)}(x) \quad . \tag{104}$$

x is a real variable and ρ a positive parameter. We assume that the coefficients $a_\nu(x)$ and $b_\mu(x)$ admit *at least* n bounded derivatives in the interval

$$\alpha \leq x \leq \beta \quad .$$

If the order n of the differential expression $N(y)$ is greater than the order m of the differential expression $M(y)$, then the differential equation (101) gives rise to a boundary layer problem for the "limiting" differential equation, i.e. the differential equation obtained by letting ρ tend to infinity in the original differential equation $L(y,\rho) = 0$. For, this limiting differential equation is

$$M(y) = 0 \quad , \tag{105}$$

and this differential equation is of lower order than (101), if

$$n > m \tag{106}$$

We shall also assume that

$$m > 0 \quad . \tag{106a}$$

Most of our results remain valid for $m = 0$. But at some points the inclusion of the case $m = 0$ would make the statement of the result rather involved. It seemed therefore preferable to exclude this case from the Main Theorem.

Together with the differential equation (101) we prescribe n boundary conditions for the function $y(x)$. The boundary conditions considered in this chapter are of the form

$$L_i(y) = \ell_i \quad , \quad (i = 1,2,\ldots,n) \tag{107}$$

with constant ℓ_i and with

$$L_i(y) = \begin{cases} y^{(\lambda_i)}(\beta) & \text{for } i = 1,2,\ldots,r \\ y^{(\tau_i)}(\alpha) & \text{for } i = r+1,\ldots,n \end{cases} , \tag{108}$$

where $x = \alpha$ and $x = \beta$ are the left and right endpoints, respectively, of the interval under consideration.*

We assume that the boundary conditions are arranged in such a way that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r$$

and

$$\tau_{r+1} > \tau_{r+2} > \cdots > \tau_n \quad .$$

This arrangement is the opposite of the customary one, but it is more practical in our case. All the numbers λ_i and τ_i are, of course, assumed to be less than n. r is the number of boundary conditions prescribed at the right endpoint. The number of boundary conditions at the left end point is then $n - r$.

One or both of the numbers λ_r and τ_n may be zero, which means that the value of the function itself is prescribed at one or both endpoints. But our theory applies also to cases in which only derivatives of the function are prescribed at the endpoints. The boundary conditions (108) contain as a special case the initial value problem. We have only to set $r = 0$, or $r = n$.

---

* We shall use throughout this paper the notation $y^{(\kappa)}(x)$ for $\dfrac{d^\kappa y}{dx^\kappa}$.

We make further the assumption

$$a_0(x) \neq 0, \quad \text{for all} \quad x \quad \text{in} \quad \alpha < x < \beta \quad , \tag{109}$$

which makes it possible for us to set

$$a_0(x) = 1 \quad , \tag{110}$$

without loss of generality.

A very essential condition for the validity of the theory that follows is that we must have also

$$b_0(x) \neq 0, \quad \text{for all} \quad x \quad \text{in} \quad \alpha < x < \beta \quad . \tag{111}$$

It is easy to see that a theory of boundary layer problems which does not assume (111) must be expected to be much more complicated. For $b_0(x)$ is the coefficient of the first term in the limiting differential equation (105). Hence, if $b_0(x)$ has zeros in $\alpha < x < \beta$, these zeros will, in general, be singular points of the solutions of the limiting differential equation.

To these assumptions we will have to add two more conditions of a rather essential nature. Since these conditions can be more easily formulated in connection with our Main Theorem we postpone their statement for a few pages.

In general, the differential equation (101) will have a unique solution $U(x,\rho)$ satisfying the boundary conditions (107). $U(x,\rho)$ depends on the value of the parameter ρ. We will be able to give a general rule which allows us to decide, for a given problem, whether

$$u(x) = \lim_{\rho\to\infty} U(x,\rho) \tag{112}$$

exists, and which are the boundary conditions satisfied by $u(x)$. We shall see also that $u(x)$, when it exists, is a solution of the limiting differential equation $M(y) = 0$.

The behavior of $U(x,\rho)$, as ρ tends to infinity, will be seen to depend, in general, on three things only:

(a) On the number $n - m$, i.e. the difference between the orders of the original and the limiting differential equation.

(b) On r, i.e. on the way in which the n boundary conditions are divided between the two end points.

(c) On the sign of the coefficient $b_0(x)$.

2. There are a great many different possible cases for our boundary layer problem. In some cases $U(x,\rho)$ converges, as $\rho \to \infty$, in some cases it diverges, and there are some special occurrences that are not covered by the Main Theorem. This accounts for the fact that the Main Theorem, although very simple to apply, is somewhat lengthy to formulate. We precede its general formulation by a few examples, in order to give, without proof, an idea of the variety of possible occurrences. In the convergent cases the boundary conditions satisfied by the limit function $u(x)$ are obtained by canceling $n - m$ of the given boundary conditions, usually taken among those involving higher orders of differentiation.

Example 1.

$$L(x,\rho) \equiv \frac{1}{\rho} y''' + x^3 y + 2 x y'' = 0$$

with the boundary conditions

$$y''(\alpha) = \ell_3 \qquad y''(\infty) = \ell_1$$
$$y'(\beta) = \ell_2 \ .$$

If, e.g., $\alpha = 1$, $\beta = 2$, then $b_0(x) > 0$ in $\alpha \leq x \leq \beta$ and the solution $U(x,\rho)$ of the problem tends to the solution of the differential equation

$$M(y) \equiv x^3 y'' + 2 x y = 0$$

satisfying the boundary conditions

$$y''(\beta) = \ell_1$$
$$y'(\beta) = \ell_2 \ ,$$

which are obtained by canceling the boundary condition given at $x = \alpha$. If $\alpha = -2$, $\beta = -1$, then $b_0(x) < 0$ and $U(x,\rho)$ tends to the solution of $M(y) = 0$ with the boundary conditions

$$y''(\alpha) = \ell_3 \qquad y'(\beta) = \ell_2$$

obtained by canceling the first boundary conditions at $x = \beta$.

But if $\alpha = -1$, $\beta = 1$, the condition (111) is no longer satisfied and our Main Theorem does not apply.

<u>Example 2</u>.

$$L(x,\rho) \equiv \frac{1}{\rho}\left(y^{(4)} + \cos x \cdot y^{(3)}\right) + xy'' + xy = 0$$

with the boundary conditions

$$y''(\alpha) = \ell_2$$

$$y'(\alpha) = \ell_3$$

$$y\ (\alpha) = \ell_4 \qquad y(\beta) = \ell_1 \ .$$

If $\alpha = -2$, $\beta = -1$, then $b_0(x) < 0$ and $U(x,\rho)$ converges to the solution of the differential equation

$$y'' + y = 0$$

satisfying the boundary conditions

$$y'(\alpha) = \ell_3$$

$$y\ (\alpha) = \ell_4$$

which are obtained by canceling the first boundary condition at each endpoint. If $\alpha > 0$, $\beta > 0$ then $b_0(x) > 0$ in $\alpha < x < \beta$, and $U(x,\rho)$ tends in general, to the solution of $y'' + y = 0$ with the boundary conditions

$$y(\alpha) = \ell_4 \qquad y(\beta) = \ell_1 \ ,$$

because in this case the general theory requires the canceling of the two boundary conditions involving the highest order of differentiation.

But if

$$\alpha = 2\pi \quad , \quad \beta = 4\pi \ ,$$

then we have an exceptional case. Because then there is no solution of $y'' + y = 0$ satisfying the boundary conditions $y(\alpha) = \ell_4$, $y(\beta) = \ell_1$, unless $\ell_1 = \ell_4 = 0$. Again, our Main Theorem does not cover these special values of α and β.

<u>Example 3</u>.

$$L(y,\rho) \equiv \frac{1}{\rho}\, y^{(4)} - xy' - y = 0$$

with the boundary conditions

$$y'''(\alpha) = \ell_4 \qquad y'''(\beta) = \ell_1$$
$$y'(\beta) = \ell_2$$
$$y(\beta) = \ell_3 \ .$$

If $\alpha < 0,\ \beta < 0,$ then $b_0(x) > 0$ and $U(x,\rho)$ *tends to a solution of*

$$xy' - y = 0$$

with the boundary condition $y(\beta) = \ell_3,$ because the Main Theorem requires the canceling of two boundary conditions at the right endpoint and of one boundary condition at the left endpoint. But if $\alpha > 0,\ \beta > 0,$ i.e. $b_0(x) < 0,$ then $U(x,\rho)$ does <u>not</u> converge at all.

§2. Statement of the Main Theorem.

3. <u>Main Theorem:</u>

Let $U(x,\rho)$ be a solution of the differential equation

$$L(y,\rho) = 0 \tag{101}$$

satisfying n boundary conditions

$$L_i(y) = \ell_i \quad , \quad i = 1,2,\ldots,n \tag{107}$$

(constant ℓ_i), where $L(y,\rho)$ if of the form

$$L(y,\rho) \equiv \frac{1}{\rho} N(y) + M(y) \tag{102}$$

with

$$N(y) = \sum_{\nu=0}^{n} a_\nu(x) y^{(n-\nu)}(x) \tag{103}$$

$$M(y) = \sum_{\mu=0}^{m} b_\mu(x) y^{(m-\mu)}(x) \tag{104}$$

and

$$L_i(y) = \begin{cases} y^{(\lambda_i)}(\beta) \ , & \text{for } i = 1,2,\ldots,r \\ y^{(\tau_i)}(\alpha) \ , & \text{for } i = r+1,\ r+2,\ldots,n \end{cases} \quad , \ \alpha < \beta \ . \tag{108}$$

We make the following assumptions:

1°. x is a real variable.

2°. ρ is a real positive parameter.

3°. The real functions $a_{\nu}(x)$ and $b_{\mu}(x)$ have at least n bounded derivatives in the interval

$$\alpha < x < \beta \ .$$

4°. $n > m > 0$.

5°. $a_0(x) = 1$.

6°. $b_0(x) \neq 0$, for all x in $\alpha < x < \beta$.

7°. $n > \lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$

$$n > \tau_{r+1} > \tau_{r+2} > \cdots > \tau_n > 0 \ .$$

Then the behavior of $U(x,\rho)$, as ρ tends to infinity, can be found by the following procedure:

First Step. Find the remainder s of the division of $n - m$ by 4.

Second Step. Find, in the table on the next page, the values of the numbers q and p for the differential equation under consideration.

$$\begin{array}{lll}
\text{(I) If} & \left.\begin{array}{l} s = 1,\ b_0 > 0 \\ \quad\text{or} \\ s = 3,\ b_0 < 0 \end{array}\right\} & \text{then } q = \dfrac{n-m+1}{2},\ p = \dfrac{n-m-1}{2} \\
\text{(II) If} & \left.\begin{array}{l} s = 1,\ b_0 < 0 \\ \quad\text{or} \\ s = 3,\ b_0 > 0 \end{array}\right\} & \text{then } q = \dfrac{n-m-1}{2},\ p = \dfrac{n-m+1}{2} \\
\text{(III) If} & \left.\begin{array}{l} s = 0,\ b_0 > 0 \\ \quad\text{or} \\ s = 2,\ b_0 < 0 \end{array}\right\} & \text{then } q = \dfrac{n-m}{2},\ \ p = \dfrac{n-m}{2} \\
\text{(IV) If} & \left.\begin{array}{l} s = 0,\ b_0 < 0 \\ \quad\text{or} \\ s = 2,\ b_0 > 0 \end{array}\right\} & \text{then } q = \dfrac{n-m-2}{2},\ p = \dfrac{n-m-2}{2}
\end{array} \qquad (113)$$

Third Step.

A) If the differential equation under consideration belongs to one of the cases (I) - (III)* of the table above, try to cancel p of the boundary conditions at the point $x = \beta$ and q of the boundary conditions at the point $x = \alpha$, going in each group of boundary conditions from those containing higher derivatives to those with lower derivatives. This is only possible, if there are enough boundary conditions on either side to be canceled.

B) If the differential equation under consideration is of the type IV, proceed first as under A). From the remaining boundary conditions cancel then those two which contain the highest order of differentiation without regard to the endpoint at which they are given. It can happen that the boundary conditions to be canceled in application of this last rule are not uniquely determined, because one would have to decide between two boundary conditions of the same order of differentiation. We shall call this last occurrence the "Case of Indetermination".

Convergent Case. If it is possible to apply the rule of the Third Step in a uniquely determined fashion, then

$$u(x) = \lim_{\rho \to \infty} U(x, \rho)$$

exists and is, in $\alpha < x < \beta$, that solution of the differential equation

$$M(y) = 0$$

which satisfies the boundary conditions not canceled in the Third Step of this rule, provided the following two conditions are satisfied:

---

* We use the circles around these numbers, writing (I), (II), (III), (IV), in order to distinguish this division into four cases from another division into two cases I, II to be introduced presently.

8°. If the boundary conditions not canceled in the Third Step of this rule are replaced by the corresponding homogeneous boundary conditions, then the problem determined by these boundary conditions and the differential equation $M(y) = 0$ has only the solution $y(x) \equiv 0$.

9°. No two of the boundary conditions canceled in the Third Step at $x = \alpha$ have orders of differentiation that are congruent modulo $n - m$, and the same is true for the boundary conditions at the right end point.

Divergent Cases. If the rule of the Third Step cannot be applied because at one of the endpoints there are not enough boundary conditions to be canceled, then $U(x,\rho)$ will, in general, not converge, as $\rho \to \infty$. The proof for the divergence given in this investigation is valid only under two assumptions analogous to 8° and 9°, which for their formula tion require an additional remark:

Fourth Step. If the rule of the Third Step cannot be applied because the number of the boundary conditions at one endpoint is smaller than the boundary conditions that would have to be canceled, then cancel all the boundary conditions on this side and so many boundary conditions on the other side (going, as before, from higher to lower order of differentiation) that m uncanceled boundary conditions remain. Then we make the assumptions:

$8^{\circ'}$. If the boundary conditions not cancelled in the fourth step of this rule are replaced by the corresponding homogeneous boundary conditions, then the problem determined by these boundary conditions and the differential equation $M(y) = 0$ has only the solution $y(x) \equiv 0$.

$9^{\circ'}$. No two of the boundary conditions canceled in the fourth step at $x = \alpha$ have orders of differentiation that are congruent modulo $n - m$, and the same is true for the boundary conditions at the right endpoint.

Conclusion in the divergent case.

1) If the rule of the third step cannot be applied because at one of the endpoints there are not enough boundary conditions to be canceled, and if conditions $8^{o'}$ and $9^{o'}$ are satisfied, then

$$\lim_{\rho\to\infty} U(x,\rho) = \pm\infty, \quad \text{for all } x \text{ in } \alpha < x < \beta .$$

2) If the rule of the third step cannot be applied because of indetermination, and if 8^o and 9^o are satisfied for each of the two possible ways of applying the cancellation rule, then $U(x,\rho)$ does not converge, as $\rho \to \infty$ but remains bounded.

4. The reader is advised to check the examples given in §1 in the light of the Main Theorem. In example 2, in particular, we discussed a case in which assumption 8^o was not satisfied. We now give an example where assumption 9^o is not satisfied:

Example 4. $n = 5$, $m = 2$, $b_0 > 0$.

$$y'''(\alpha) = \ell_3 \qquad y^{(4)}(\beta) = \ell_1$$
$$y'\ (\alpha) = \ell_4 \qquad y'\ (\beta) = \ell_2$$
$$y\ (\alpha) = \ell_5$$

Here $n-m = 3$, hence $s = 3$. From table (113) we find $q = 1$, $p = 2$.

The two boundary conditions that are to be canceled at $x = \beta$ have the orders of differentiation 4 and 1. But $4 \equiv 1 \pmod{n-m}$, in this case. This means, assumption 9^o is not satisfied, and the Main Theorem does not apply. However, if $b_0 < 0$, then 9^o is satisfied, and we can be sure of the convergence of $U(x,\rho)$.

It is an open question whether $U(x,\rho)$ can converge even if 9^o is not satisfied. It seems unlikely to the author that the Main Theorem remains valid in those cases.

The next example is of the type which we have called the case of indetermination.

Example 5. $n = 3$, $m = 1$, $b_0 > 0$,

$$y'(\alpha) = \ell_2 \qquad\qquad y(\beta) = \ell_1$$

$$y\,(\alpha) = \ell_3$$

Here, $n-m = 2$ and therefore $s = 2$. Table (113) shows that this is the case IV, and that $p = q = 0$. The rules of the Main Theorem would require the canceling of the two boundary conditions involving the highest derivatives. This cannot be done in a uniquely determined way, since $y(\beta) = \ell_1$ just as well as $y(\alpha) = \ell_3$ might be canceled in addition to $y'(\alpha) = \ell_2$. Hence, this is the case of indetermination, and $U(x,\rho)$ does not converge.

The rest of this chapter is devoted to the proof of the Main Theorem.

§3. Asymptotic Solution of the Differential Equation $L(x,\rho) = 0$.

5. As pointed out in the introduction the principal tool of our proof of the Main Theorem is the theory of asymptotic solution of differential equations involving a parameter. We begin by defining what we shall understand by asymptotic equality in this investigation.

Definition: The functions $f(x,\rho)$ and $g(x,\rho)$ are said to be asymptotically equal in an interval $\alpha \leq x \leq \beta$, if

$$f(x,\rho) = g(x,\rho) + \frac{E(x,\rho)}{\rho^a} .$$

Here $a > 0$, (but not necessarily an integer), and $E(x,\rho)$ is a function such that there is a positive real number R so that $|E(x,\rho)|$ is uniformly bounded for $\alpha \leq x \leq \beta$, and $\rho > R$.

If a function $f(x,\rho)$ is asymptotically equal to a function $F(x)$ independent of ρ, we shall write

$$f(x,\rho) = [F(x)] .$$

Note that the symbol $[F(x)]$ does not describe the function $f(x,\rho)$ uniquely. It is <u>not</u> correct to conclude from

$$f_1(x,\rho) = [F(x)]$$

and

$$f_2(x,\rho) = [F(x)]$$

that

$$f_1(x,\rho) = f_2(x,\rho) \quad .$$

6. Using Noaillon's method the following theorem can be proved.

<u>Theorem 1</u>: If the assumptions 1° - 6° of the Main Theorem are satisfied, then the differential equation (101) admits a complete set of n linearly independent solutions of the form

$$U_\nu(x,\rho) = \begin{cases} e^{\sigma \int_\alpha^x \phi_\nu(\xi)d\xi}[\eta(x)], \ (\nu = 1,2,\ldots,n-m) & (114) \\ \\ [u(x)_{\nu-n+m}] \quad , \quad (\nu = n-m+1,\ n-m+2,\ldots,n) & (115) \end{cases}$$

Here we are using the following abbreviations

1) $\sigma = |\rho^{1/n-m}|$ (116)

2) $\phi_1(x),\ \phi_2(x),\ldots,\phi_{n-m}(x)$

n-m values of

$$(-b_0(x))^{1/n-m}$$

arranged in such a way that

$$Re(\phi_1) > Re(\phi_2) > \cdots > Re(\phi_{n-m}) \quad . \tag{117}$$

3)

$$\eta(x) = e^{-\int_\alpha^x \frac{a_1(\xi)b_0(\xi) - b_1(\xi)}{b_0(\xi)(n-m)}d\xi} \tag{118}$$

4) The functions $u_\mu(x)$, $(\mu = 1,2,\ldots,m)$ are any m solutions of the differential equation

$$M(y) = 0 \qquad 1o5)$$

forming a complete linearly independent system of such solutions.

The equations (114) and (115) may be formally differentiated at least $n-1$ times, i.e.,

$$U_\nu^{(i)}(x,\rho) = \begin{cases} \sigma^i e^{\sigma \int_\alpha^x \varphi_\nu(\xi)d\xi} [\varphi_\nu^i(x) + \eta(x)] \; , \; (\nu = 1,2,\ldots,n-m) & (119) \\ [u_{\nu-n+m}^{(i)}) \; , \; (\nu = n-m+1, n-m+2,\ldots,n) & (120) \end{cases}$$

for $i = 0,1,\ldots,n-1$.

7. Remarks.

As roots of one and the same real function, the complex functions $\varphi_\nu(x)$ are of a particularly simple structure. If $b_0(x) < 0$, then the $\varphi_\nu(x)$ are obtained by multiplying the (n-m)-th roots of unity by the factor

$$\left|\sqrt[n-m]{-b_0(x)}\right|$$

A similar relation holds when $b_0(x) > 0$. More precisely: Set

$$k(x) = \begin{cases} \left|b_0(x)^{1/_{n-m}}\right| \; , \; \text{if } b_0 < 0 & (121) \\ \left|b_0(x)^{1/_{n-m}}\right| e^{\frac{\pi i}{n-m}} \; , \; \text{if } b_0 > 0 & (122) \end{cases}$$

and let

$$\varepsilon = e^{\frac{2\pi i}{n-m}} \; . \qquad (123)$$

Then the $\varphi_\nu(x)$ are, - in different order, - equal to

$$k, k\varepsilon, k\varepsilon^2, \ldots\ldots, k\varepsilon^{n-m-1} \; . \qquad (124)$$

Furthermore, since the $\varphi_\nu(x)$ are the roots of a real number, the sequence of the $\varphi_\nu(x)$ consists of pairs of conjugate numbers. Hence, in (117) the ">" and the "=" signs alternate.

§4. Outline of the Asymptotic Solution of the Boundary Value Problem

8. We try to represent the solution of the boundary value problem defined by (101) and (108) in the form

$$U(x,\rho) = \sum_{\nu=1}^{n} c_\nu(\rho)\, U_\nu(x,\rho) \quad . \tag{125}$$

If such a solution exists, then the function $c_\nu(\rho)$ are solutions of the system of linear equations

$$\sum_{\nu=1}^{n} c_\nu(\rho)\cdot L_i(U_\nu) = \ell_i, \ (i = 1,2,\ldots,n) \quad . \tag{126}$$

Using (108) and substituting for the $U(x,\rho)$ the expressions (114) we find, for $\nu = 1,2,\ldots,n-m$,

$$L_i(U_\nu) = \begin{cases} \sigma^{\lambda_i} e^{\sigma w_\nu} [\varphi_\nu^{\lambda_i}(\beta)\eta(\beta)], & (i = 1,2,\ldots,r) \\ \sigma^{\tau_i} \qquad [\varphi_\nu^{\tau_i}(\alpha)\eta(\alpha)], & (i = r+1,\ r+2,\ldots,n) \end{cases} , \ (\nu = 1,2,\ldots,n-m) \tag{127}$$

where

$$w_\nu = \int_\alpha^\beta \varphi_\nu(\xi)d\xi \quad , \quad (\nu = 1,2,\ldots,n-m) \quad . \tag{128}$$

For $\nu = n-m+1,\ldots,n$ we have, because of (115),

$$L_i(U_\nu) = [L_i(u_{\nu-n+m})] \quad . \tag{129}$$

From the inequalities (117) it follows that similar inequalities hold for the quantities w_ν $(\nu = 1,2,\ldots,n-m)$, i.e.

$$\mathrm{Re}(w_1) > \mathrm{Re}(w_2) > \cdots > \mathrm{Re}(w_{n-m}) \quad . \tag{130}$$

In order to find the $c_\nu(\rho)$ of (125) from (126) we have to calculate the determinant

$$\Delta(\rho) = \begin{vmatrix} L_1(U_1) & L_1(U_2) & \cdots & L_1(U_n) \\ L_2(U_1) & L_2(U_2) & \cdots & L_2(U_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ L_n(U_1) & L_n(U_2) & \cdots & L_n(U_n) \end{vmatrix} \tag{131}$$

and the determinants $\Delta_\nu(\rho)$, $(\nu = 1,2,\ldots,n)$, obtained by replacing the ν-th column of $\Delta(\rho)$ by the column

$$\begin{matrix} \ell_1 \\ \ell_2 \\ \cdot \\ \cdot \\ \cdot \\ \ell_n \end{matrix} \quad .$$

The coefficients $c_\nu(\rho)$ are then given by

$$c_\nu(\rho) = \frac{\Delta_\nu(\rho)}{\Delta(\rho)} \quad , \quad (\nu = 1,2,\ldots,n) \tag{1.32}$$

and, if we substitute (132) in (125) we obtain the form

$$U(x,\rho) = \sum_{\nu=1}^{n} \frac{\Delta_\nu(\rho)}{\Delta(\rho)} U_\nu(x,\rho) \tag{133}$$

for the solution $U(x,\rho)$ of our boundary value problem.

9. Our aim is now to calculate, with the help of the asymptotic expressions (114), (115) and (127), (129) the asymptotic value of the right number of (133). The first and most important part of that calculation consists in finding the asymptotic value of the determinant $\Delta(\rho)$. The asymptotic calculation of the $\Delta_\nu(\rho)$ does not offer new difficulties.

§5. The Asymptotic Value of $\Delta(\rho)$

10. As a consequence of (127) and (129) all the terms of the expansion of $\Delta(\rho)$ are obviously of the form

$$[k]\,\sigma^S\, e^{\sigma V}$$

with real S and real or complex V and k.

Definition: Two expressions

$$K_1 = [k_1]\,\sigma^{S_1}\, e^{\sigma V_1}$$

$$K_2 = [k_2]\,\sigma^{S_2}\, e^{\sigma V_2}$$

with

$$k_1 \neq 0, \quad k_2 \neq 0$$

and real S_1 and S_2 will be said to be of equal order of magnitude if

$$\mathrm{Re}(V_1) = \mathrm{Re}(V_2)$$

and

$$S_1 = S_2 \ .$$

If

$$\mathrm{Re}(V_1) > \mathrm{Re}(V_2)$$

or

$$\mathrm{Re}(V_1) = \mathrm{Re}(V_2), \text{ but } S_1 > S_2$$

then K_1 is said to be of higher order of magnitude than K_2, and vice versa.

If K_1 is of higher order of magnitude than K_2, then we can obviously write

$$K_1 + K_2 = [k_1]\,\sigma^{S_1}\, e^{\sigma V_1} \ .$$

The sum

$$\sum_{\omega} [k_{\omega}]\ \sigma^{S} e^{V_{\omega}} \tag{134}$$

of all the terms of highest order in $\Delta(\rho)$ is the asymptotic expression of $\Delta(\rho)$ for large ρ, unless all the V_{ω} are alike and

$$\sum_{\omega} k_{\omega} = 0 \quad .$$

In this latter case (134) reduced to

$$[0]\ \sigma^{S} e^{V_1}$$

and an asymptotic calculation of $\Delta(\rho)$ would have to take into account terms of lower order of magnitude in the expansion of $\Delta(\rho)$, as well as the later terms in the asymptotic solutions of our differential equation. The exclusion of this exceptional case from our theory will compel us to introduce the conditions 8° and 9° of the Main Theorem.

11. If the values (127) and (129), for the $L_i(U_{\nu})$ are substituted into the expression (131) and $\Delta(\rho)$, it is seen that the last m columns of the determinant form the matrix of n rows and m columns

$$\begin{pmatrix} [L_1(u_1)] & [L_1(u_2)] & \cdots & [L_1(u_m)] \\ [L_2(u_1)] & [L_2(u_2)] & \cdots & [L_2(u_m)] \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ [L_n(u_1)] & [L_n(u_2)] & \cdots & [L_n(u_m)] \end{pmatrix} \quad . \tag{135}$$

All the minors of this matrix have an order of magnitude not greater than that of 1.

The elements of the first $n-m$ columns of $\Delta(\rho)$ are given by (127). In order to find the asymptotic value of $\Delta(\rho)$ we expand $\Delta(\rho)$ in terms of its $n-m$ first columns and investigate the order of magnitude of the minors in this expansion.

12. Lemma 1: Let $D(\rho)$ be that minor of the determinant $\Delta(\rho)$ which is formed by the first $n-m$ columns of $\Delta(\rho)$ and by its

$$i_1^{th}, i_2^{th}, \ldots, i_{n-m}^{th}$$

rows, with

$$i_1 < i_2 < \cdots < i_{n-m} .$$

If then the first h of the numbers i are less than or equal to r, then the order of magnitude of $D(\rho)$ is not greater than that of

$$\sigma^T e^{\sigma \sum_{\nu=1}^{h} w_\nu} \tag{136}$$

where

$$T = \sum_{\nu=1}^{h} \lambda_{i_\nu} + \sum_{\mu=h+1}^{n-m} \tau_{i_\mu} . \tag{137}$$

Proof: From (127) and (131) we see that each of the first h rows of $D(\rho)$ contains the factor $\eta(\beta)$, *while each of the remaining rows contains the factor* $\eta(\alpha)$. Hence, we can factor in $D(\rho)$ the expression

$$\eta^h(\beta)\ \eta^{n-m-h}(\alpha) .$$

Furthermore, we see, that the first row of $D(\rho)$ contains the factor $\sigma^{\lambda_{i_1}}$, the second row the factor $\sigma^{\lambda_{i_2}}$, etc., and finally the h-th row the factor $\sigma^{\lambda_{i_h}}$. Similarly, we can factor in the remaining rows of $D(\rho)$ the expressions

$$\sigma^{\tau_{i_{h+1}}}, \sigma^{\tau_{i_{h+2}}}, \ldots, \sigma^{\tau_{i_{n-m}}} ,$$

respectively. Altogether we can factor in $D(\rho)$ the expression

$$\eta^h(\beta)\ \eta^{n-m-h}(\alpha)\ \sigma^T ,$$

where T has the value of (137). Then $D(\rho)$ can be written in the form

$$D(\rho) = \eta^{h}(\beta)\,\eta^{n-m-h}(\alpha)\,\sigma^{T} \begin{vmatrix} e^{\sigma w_1}[\varphi_1^{\lambda_{i_1}}(\beta)] & \cdots\cdots & e^{\sigma w_{n-m}}[\varphi_{n-m}^{\lambda_{i_1}}(\beta)] \\ \cdots & \cdots & \cdots \\ e^{\sigma w_1}[\varphi_1^{\lambda_{i_h}}(\beta)] & \cdots\cdots & e^{\sigma w_{n-m}}[\varphi_{n-m}^{\lambda_{i_h}}(\beta)] \\ [\varphi_1^{\tau_{i_{h+1}}}(\alpha)] & \cdots\cdots & [\varphi_{n-m}^{\tau_{i_{h+1}}}] \\ \cdots & \cdots & \cdots \\ [\varphi_1^{\tau_{i_{n-m}}}(\alpha)] & \cdots\cdots & [\varphi_{n-m}^{\tau_{i_{n-m}}}(\alpha)] \end{vmatrix} \tag{138}$$

Now we expand the remaining determinant with respect to the minors of its first h rows. Each term of this expansion contains an exponential factor, and none of these exponential factors is of greater order of magnitude than the one originating from the minor formed by the first $n-m$ rows and columns of the determinant of (138). For the exponential factor in this term is

$$e^{\sigma \sum_{\nu=1}^{h} w_\nu},$$

and because of (130) no sum of h w_ν's has a greater real part than $\sum_{\nu=1}^{h} w_\nu$. Hence, the order of magnitude of $D(\rho)$ is not greater than that of

$$\sigma^{T} e^{\sigma \sum_{\nu=1}^{h} w_\nu}.$$

It may be less, for we have to take into account the possibility that

$$D(\rho) = [0]\,\sigma^{T} e^{\sigma \sum_{\nu=1}^{h} w_\nu} .$$

Remark: If $Re(w_{h+1}) < Re(w_h)$, then there is only one term of maximal order of magnitude in the expansion of the determinant of (138) with respect to its first h rows. But if we have $Re(w_{h+1}) = Re(w_h)$ (compare section 7 and the definition of the w_ν, formula (128)), then

$$Re\Big(\sum_{\nu=1}^{h} w_\nu\Big) = Re\Big(\sum_{\nu=1}^{h-1} w_\nu + w_{h+1}\Big) ,$$

and we have therefore a second term of maximal order in the expansion of the determinant of (140) with respect to its first h rows. But, clearly, these two terms cannot cancel, since

$$w_h \neq w_{h+1} .$$

13. The question of finding among all the minors of the first $n-m$ columns of $\Delta(\rho)$ the one of highest order reduces now to the two questions:

(a) which selection of $n-m$ rows in $\Delta(\rho)$ leads to an expression (136) of highest order, and,

(b) when does the minor corresponding to this selection actually have the order indicated by (136).

In answering the first question several cases are to be distinguished. For this distinction the number of $\varphi_\nu(x)$'s which have a positive real part plays an essential role. Let us call this number p. From the definition of the φ_ν's it is clear that p depends on the sign of $b_0(x)$ and on the remainder of the division of $n-m$ by 4. A simple calculation, which we omit here shows that p has the values indicated in the table (113).

We distinguish the following cases:

I. $Re(\varphi_\nu) \neq 0$, for all $\nu = 1,2,\ldots,n-m$

A) $r-p < 0$

B) $0 < r-p < m$

C) $r-p > m$

II. $Re(\varphi_\nu) = 0$, for some ν .

A) $r-p < 0$

B) $0 < r-p < m+2$

C) $r-p > m+2$.

<u>Remarks</u>: Remembering the definition of the ϕ_ν we see immediately that case II of (139) occurs only when

$$
\begin{aligned}
&n-m \equiv 0 \pmod 4 \quad \text{and} \quad b_0 < 0 \\
&\text{or} \\
&n-m \equiv 2 \pmod 4 \quad \text{and} \quad b_0 > 0 \quad .
\end{aligned}
\tag{140}
$$

Case II is thus seen to be equivalent with the case IV of table (113). In this case IV there are always exactly <u>two</u> functions φ_ν with vanishing real parts. In the arrangement of (117) these are

$$\varphi_{p+1}(x) \quad \text{and} \quad \varphi_{p+2}(x) \quad .$$

Case I of table (139) corresponds to the cases I - III of table (113). In these cases the number q of table (113) is equal to $n-m-p$. It follows, therefore, that the condition IC of (139) can be re-written in the form

$$(n-r) - q < 0 \quad .$$

In case IV of (113), which we have seen to be equivalent to case II of (139), we see from (113) that

$$q = n-m-p-2 \quad ,$$

and therefore IIC is also equivalent to (141). Hence, we can say, that in case I as well as in case II of (139)

A) is the case in which the canceling rule of the Main Theorem cannot be followed because there are not enough boundary conditions at $x = \beta$.

B) is the case in which the canceling rule <u>can</u> be followed.

C) is the case in which the canceling rule cannot be followed, because there are not enough boundary conditions at $x = \alpha$.

14. We treat the case I of (139) first. According as to whether we are in the case IA, IB, or IC, we consider then the minor $D_a(\rho)$, $D_b(\rho)$, $D_0(\rho)$ formed by the $n-m$ first columns of the determinant $\Delta(\rho)$ and by the rows

$1, 2, \ldots\ldots, n-m$.			(case A), (142a)
$1, 2, \ldots p$;	$r+1, r+2, \ldots\ldots, r+n-m-p$.		(case B), (142b)
$1, 3, \ldots\ldots, r-m$;	$r+1, r+2, \ldots\ldots, n$.		(case C), (142c)

respectively. Using lemma 1 we shall show that in each case the minor thus defined is - in general - of the highest possible order of magnitude among all the minors of the first $n-m$ columns of $\Delta(\rho)$.

Case A) The expression (136) has the highest possible order, if the rows of the minor are chosen in such a way that $\sum_{\nu=1}^{h} w_\nu$ has the greatest possible real part. In case IA this means that we have to choose $h = r$. For h can, - by definition, - not be greater than r and, on the other hand, all w_ν with $\nu \leq r$ have positive real parts in consequence of the condition $r-p<0$. In order to make T in (136) as great as possible we have then to choose for the remaining $n-m-r$ rows of the minor those for which $\sum_{\mu=h+1}^{n-m} \tau_{i_\mu}$ is greatest. Since the τ_i are arranged in order of decreasing size, this is the case, if we choose the rows $r+1, r+2, \ldots, n-m$. This is exactly what we have done in (142a).

Case B) Here, taking $h = r$ would not make the real part of $\sum_{\nu=1}^{h} w_\nu$ a maximum because we have $r > p$ and the sume would therefore include w_ν's with negative real part. Instead, we have to take $h = p$ including thus all the w_ν's with positive real part and only those. (137) shows then that taking the <u>first</u> p rows of $\Delta(\rho)$ gives the greatest contribution to the exponent T. For the remaining $n-m-p$ rows we take the rows $r+1, r+2, \ldots, r+n-m-p$ in order to make the second sum of the right number of (137) a maximum. This is possible because, in consequence of (139), $r+n-m-p \leq n$, in this case.

Case C) Here we reason as in case B. But since $r+n-m-p > n$ in this case, taking the rows $r+1, r+2, \ldots, n$ in addition to the rows $1,2, \ldots, p$ would not be enough to have $n-m$ rows altogether. We must therefore choose $h = r-m$, in order to have $n-m$ rows, and, as before, we see that T is greatest, if we take the first $r-m$ rows of $\Delta(\rho)$.

Comparing (142b) with the third step of the rule of the Main Theorem we see that the numbers of (142b) are just the subscripts of the boundary conditions canceled in application of the Main Theorem in the cases I - III, when the canceling is possible. Similarly, (142a) and (142c) contains the subscripts of the boundry conditions canceled in the fourth step of the Main Theorem in the cases I - III. The reason for this fact will appear in the course of our proof.

15. The considerations of section 14 are not sufficient to prove that the minors $D_a(\rho)$, $D_b(\rho)$, $D_c(\rho)$, respectively, really do have higher order than all the others (compare section 12). To investigate this question let us calculate $D_a(\rho)$, $D_b(\rho)$, $D_c(\rho)$ explicitly. Since we are most interested in case IB, which we shall see to be the convergent case, we discuss $D_b(\rho)$ first.

In order to calculate $D_b(\rho)$ we write $D_b(\rho)$ in the form (138). This means that we interpret the numbers

$$i_1, i_2, \ldots, i_{n-m}$$

of (138) as being the numbers (142b), in the same order. Hence, the number h of (138) is in this case equal to p. From the definition of p and the w_ν it follows that

$$\mathrm{Re}(w_{p+1}) < \mathrm{Re}(w_p) \quad .$$

The term originating from the minor of the first p columns in the expansion of the determinant (138) in terms of its first p rows is therefore the only term of maximal order of magnitude in this expansion. We obtain therefore

$$D_b(\rho) = [\eta^p(\beta)\eta^{n-m-p}(\alpha) \cdot A_b \cdot B_b]\, \sigma^{T_b}\, e^{\sigma \sum_{\nu=1}^{p} w_\nu} \tag{143}$$

with

$$\tau_b = \sum_{\nu=1}^{p} \lambda_\nu + \sum_{\mu=p+1}^{r+n-m-p} \tau_\mu \tag{144}$$

and

$$A_b = \pm \begin{vmatrix} \varphi_1^{\lambda_1}(\beta) \cdots \varphi_p^{\lambda_1}(\beta) \\ \cdots\cdots\cdots \\ \varphi_1^{\lambda_p}(\beta) \cdots \varphi_p^{\lambda_p}(\beta) \end{vmatrix} \tag{145}$$

$$B_b = \pm \begin{vmatrix} \varphi_{p+1}^{\tau_{r+1}}(\alpha) \cdots\cdots \varphi_{n-m}^{\tau_{r+1}}(\alpha) \\ \cdots\cdots\cdots\cdots \\ \varphi_{p+1}^{\tau_{r+n-m-p}}(\alpha) \cdots \varphi_{n-m}^{\tau_{r+n-m-p}}(\alpha) \end{vmatrix} \tag{146}$$

(134) is valid also for $p = n-m$ and for $p = 0$, if we define $A_b = 1$, for $p = 0$, and $B_b = 1$, for $p = n-m$. $\eta(x)$ is an exponential function, hence $\eta(\beta) \neq 0$, $\eta(\alpha) \neq 0$. $D_b(\rho)$ has therefore the order of magnitude of

$$\sigma^{\tau_b} \cdot e^{\sigma \sum_{\nu=1}^{p} w_n}$$

if $A_b \neq 0$ and $B_b \neq 0$.

<u>Lemma 2</u>: $A_b \neq 0$ if and only if

$$\lambda_i \not\equiv \lambda_j, \ (\text{mod } n-m); \qquad (i,j = 0,1,\ldots,p) \tag{147}$$

and $B_b \neq 0$ if and only if

$$\tau_n \not\equiv \tau_\ell, \ (\text{mod } n-m); \qquad (k,\ell = r+1, \ldots, r+n-m-p) \ . \tag{148}$$

<u>Proof</u>: The statement is trivial, as far as A_b is concerned, if $p = 0$. For $p > 0$ we remember from §3, section 7, that, with the notations used there, the $\varphi_\nu(x)$ are, in different order, equal to

$$k(x),\ k(x)\varepsilon,\ k(x)\varepsilon^2,\ \cdots,\ k(x)\varepsilon^{n-m-1}\ . \tag{124}$$

In the order of (124) the φ_ν are represented in the complex plane by a sequence of <u>successive</u> points on the circle of radius $|k|$. It is easy to see that the numbers $\varphi_1(\beta)$, $\varphi_2(\beta),\ \cdots,\ \varphi_p(\beta)$ are then, in different order, equal to the numbers

$$k(\beta)\varepsilon^{t+1},\ k(\beta)\varepsilon^{t+2},\ \cdots,\ k(\beta)\varepsilon^{t+p} \tag{149}$$

where t is a certain integer which is only determined modulo $n-m$. Substituting the expressions (149) into (147) we find

$$A_b = \pm\, k(\beta)^{\sum_{\nu=1}^{p}\lambda_\nu} \begin{vmatrix} \varepsilon^{(t+1)\lambda_1} & \cdots & \varepsilon^{(t+p)\lambda_1} \\ \cdots & \cdots & \cdots \\ \varepsilon^{(t+1)\lambda_p} & \cdots & \varepsilon^{(t+p)\lambda_p} \end{vmatrix} . \tag{150}$$

Now we set

$$\zeta_\nu = \varepsilon^{\lambda_\nu}\ , \quad (\nu = 1,2,\ldots,p)\ . \tag{151}$$

This allows us to write (150) in the form

$$A_b = \pm\, k(\beta)^{\sum_{\nu=1}^{p}\lambda_\nu} \begin{vmatrix} \zeta_1^{t+1} & \cdots & \zeta_1^{t+p} \\ \\ \zeta_1^{t+1} & \cdots & \zeta_p^{t+p} \end{vmatrix}$$

or

$$A_b = \pm\, k(\beta)^{\sum_{\nu=1}^{p}\lambda_\nu} \cdot (\zeta_1, \zeta_2 \cdots \zeta_p)^t \cdot V(\zeta_1, \zeta_2, \cdots, \zeta_p) \tag{152}$$

where $V(\zeta_1,\zeta_2,\cdots,\zeta_p)$ is the Vandermonde determinant of $\zeta_1,\zeta_2,\cdots,\zeta_p$. Since the Vandermonde determinant vanishes if and only if two of its rows are equal, A_b is zero if and only if some of the ζ_ν are alike. But from (151) we see that

$$\zeta_i = \zeta_j$$

means

$$\varepsilon^{\lambda_i} = \varepsilon^{\lambda_j}$$

or, because of the definition of ε in formula (123),

$$\lambda_i \equiv \lambda_j \quad , \quad (\text{mod. } n-m) \quad .$$

This proves the part of lemma 2b that is concerned with A_b. The proof for B_b is exactly analogous, and is therefore left to the reader.

The reader will readily see that similar results hold for $D_a(\rho)$ and $D_c(\rho)$. The only real difference in the reasoning comes from the fact that in these cases there may be <u>two</u> terms of maximal order in the expansion of (140). But since these terms cannot cancel, this does not essentially affect our argument. We restrict ourselves to stating the results in these cases:

$$D_a(\rho) = [Q_a]\,\sigma^{T_a}\, e^{\sigma \sum_{\nu=1}^{r} w_\nu} \quad , \tag{153}$$

$$D_c(\rho) = [Q_c]\,\sigma^{T_c}\, e^{\sigma \sum_{\nu=1}^{r-m} w_\nu} \quad , \tag{154}$$

where

$$T_a = \sum_{\nu=1}^{r} \lambda_\nu + \sum_{\mu=r+1}^{n-m} \tau_\mu \tag{155}$$

$$T_c = \sum_{\nu=1}^{r-m} \lambda_\nu + \sum_{\mu=r+1}^{n} \tau_\mu \tag{156}$$

and Q_a and Q_c are two constants, with respect to which the following two lemmas hold:

Lemma 2a: $Q_a \neq 0$ if

$$\left.\begin{array}{l} \lambda_i \not\equiv \lambda_j, \quad \text{mod } n-m, \; (i,j = 1,2,\dots,r) \\ \\ \tau_k \not\equiv \tau_\ell, \quad \text{mod } n-m, \; (k,\ell = r+1,\dots,n-m) \end{array}\right\} \quad . \tag{157}$$

and

Lemma 2c: $Q_c \neq 0$ if

$$\left.\begin{array}{l} \lambda_i \not\equiv \lambda_j, \quad \text{mod } n-m, \; (i,j = 1,2,\dots,r-m) \\ \tau_k \not\equiv \tau_\ell, \quad \text{mod } n-m, \; (k,\ell = r+1,\dots,n) \end{array}\right\} \quad . \tag{158}$$

Remark: From these lemmas and the remark at the end of section 14 we recognize that the conditions (147) and (148) are equivalent with the assumptions 9° of the Main Theorem while (157) and (158) are equivalent with 9°'. Note, however, that our reasoning so far does not cover the case II of (139), which we have seen to be the same as the case IV of (113).

16. The cofactor of the minor $D_v(\rho)$ in $\Delta(\rho)$ is that minor of the matrix (135) which is formed by the m rows of $\Delta(\rho)$ not contained in $D_v(\rho)$. Let

$$\delta_b = \begin{vmatrix} L_{p+1}(u_1) & \cdots & L_{p+1}(u_m) \\ \cdots & \cdots & \cdots \\ L_r(u_1) & \cdots & L_r(u_m) \\ L_{r+n-m-p+1}(u_1) & \cdots & L_{r+n-m-p+1}(u_m) \\ \cdots & \cdots & \cdots \\ L_n(u_1) & \cdots & L_n(u_m) \end{vmatrix} \tag{159}$$

then the cofactor of $D_b(\rho)$ is of the form

$$\pm [\delta_b] \quad .$$

We now introduce the assumption

$$\delta_b \neq 0, \quad \text{in case IB} \quad . \tag{160}$$

This condition will be seen later to be equivalent with the assumption 8° of the Main Theorem in the cases I - III. Then the term $\pm D_b(\rho)\,[\delta_b]$ is the term of greatest

order of magnitude in the expansion of $\Delta(\rho)$ in terms of its first $n-m$ columns. Hence, (143) gives

$$\Delta(\rho) = [Q_b \cdot \delta_b]\, \sigma^{T_b}\, e^{\sigma \sum_{\nu=1}^{p} w_\nu} \quad , \text{ in case IB} \quad , \tag{161}$$

where

$$Q_b = \pm \pi^p(\beta) \cdot \pi(\alpha)^{n-m-p} \cdot A_b \cdot B_b \neq 0 \quad . \tag{162}$$

Similarly, if we define δ_a and δ_c as the limits - as $\rho \to \infty$, - of the matrix (135) formed with the rows of $D(\rho)$ not occurring in $D_a(\rho)$ and $D_c(\rho)$, respectively, and if we introduce the assumptions

$$\delta_a \neq 0, \text{ in case IA} \quad , \tag{163}$$

$$\delta_c \neq 0, \text{ in case IB} \quad , \tag{164}$$

then we have, in analogy with (161)

$$\Delta(\rho) = [Q_a \cdot \delta_a] \cdot \sigma^{T_a}\, e^{\sigma \sum_{\nu=1}^{r} w_\nu} \quad , \text{ in case IA} \quad , \tag{165}$$

$$\Delta(\rho) = [Q_c \cdot \delta_c] \cdot \sigma^{T_c}\, e^{\sigma \sum_{\nu=1}^{r-m} w_\nu} \quad , \text{ in case IC} \quad . \tag{166}$$

This finishes the asymptotic calculation of $\Delta(\rho)$ in the case I of (139).

§6. The Asymptotic Value of the Solution of the Boundary Value Problem in the Case IB.

17. The method used in §5 for the calculation of $\Delta(\rho)$ can also be applied to the determinants $\Delta_\nu(\rho)$. (For the definition of $\Delta_\nu(\rho)$, see §4, section 8).

For $\nu > n-m$, $\Delta_\nu(\rho)$ is distinguished from $\Delta(\rho)$ only in one of the last m columns. In this case all the considerations of §5 remain valid for $\Delta_\nu(\rho)$, if the determinants δ_a, δ_b, δ_c are replaced by the determinants $\delta_{a\nu}$, $\delta_{b\nu}$, $\delta_{c\nu}$ obtained by replacing the

i-(n-m)th column of δ_a, δ_b, δ_c by the corresponding ℓ_i, i.e., by

$$\begin{array}{ccccc} \ell_{n-m+1} & & \ell_{p+1} & & \ell_{r-m+1} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \ell_r & \text{or} & \cdot \\ \cdot & & \ell_{r+n-m-p+1} & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \ell_n \; , & & \ell_n & & \ell_r \end{array}$$

respectively. Hence, we obtain, in case IB and for $\nu > n-m$,

$$\Delta_\nu(\rho) = [\Omega_b \cdot \delta_{b\nu}] \cdot \sigma^{T_b} e^{\sigma \sum_{\nu=1}^{p} w_\nu} , \; (\nu > n-m; \text{ case IB}) \tag{166}$$

and analogous formulas hold in the two other cases.

Hence

$$\frac{\Delta_\nu(\rho)}{\Delta\,(\rho)} = \left[\frac{\delta_{b\nu}}{\delta_b}\right] \; (\nu > n-m); \text{ case IB)} \quad , \tag{167}$$

and similarly in the two other cases.

18. For $\nu < n-m$ the column of $\Delta(\rho)$ that must be replaced by the ℓ_i's in order to obtain $\Delta_\nu(\rho)$ changes the structure of the determinant somewhat. But if we place this ν-th column behind all the others (and change the sign of the determinant, if necessary), then we obtain a determinant very similar to $\Delta(\rho)$. The only essential difference is that m has to be replaced by $m' = m+1$ (and therefore $n-m$ by $n-m-1$) and that w_ν must be omitted from the sequence

$w_1, w_2, \ldots, w_{n-m}$.

Again we treat the case IB (the convergent case, as we shall see) first, and since we are more interested in $\frac{\Delta_\nu(\rho)}{\Delta(\rho)}$ than in $\Delta_\nu(\rho)$ itself, we state the result in the following form:

<u>Lemma 3</u>: In case IB, i.e., if $0 < r-p < m$, we have

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)} = \begin{cases} \sigma^{-\lambda_p} e^{-\sigma w_\nu}[\pi_\nu]; & \nu < p \\ \sigma^{-\tau_{r+n-m-p}}[\pi_\nu]; & p < \nu < n-m \end{cases}, \tag{168}$$

where the constants π_ν depend on the values of the ℓ_i's and are not necessarily different from zero.

<u>Proof</u>:

1.) $\nu < p$. In this case the w_ν to be omitted has a positive real part. Hence, the formulas for $\Delta(\rho)$ can be applied to $\Delta_\nu(\rho)$ if we replace

$$p \text{ by } p' = p-1$$

$$m \text{ by } m' = m+1$$

and the sequence

$$w_1, w_2, \ldots, w_{n-m}$$

by the sequence

$$w_1', w_2', \ldots, w_{n-m-1}'$$

identical with the first one except for the omission of the term w_ν. Since we have

$$0 < r-p' < m' ,$$

the formula to be used in (161). We obtain

$$\Delta_\nu(\rho) = [\Omega_{b\nu} \cdot \delta_{b\nu}] \cdot \sigma^{T_{b\nu}} e^{\sigma \sum_{\omega=1}^{p'} w_\omega'}$$

where

$$T_{b\nu} = \sum_{\omega=1}^{p'} \lambda_\omega + \sum_{\mu=r+1}^{r+n-m'-p'} \tau_\mu \ .$$

Note that the reasoning used for the proof of $\Omega_b \neq 0$ does no longer hold for $\Omega_{b\nu} \cdot \delta_{b\nu}$ is a determinant depending on the ℓ_i. It is defined as the cofactor of the leading minor in the expansion of $\Delta_\nu(\rho)$ with respect to its first $n-m-1$ columns. $\delta_{b\nu}$ is a determinant of $m+1$ rows.

From the definition of p', m', w'_ω it follows then that

$$\sum_{\omega=1}^{p'} w'_\omega = \sum_{\omega=1}^{p} w_\omega - w_\nu$$

and

$$T_{b\nu} = \sum_{\omega=1}^{p-1} \lambda_\omega + \sum_{\mu=r+1}^{r+n-m-p} \tau_\mu \ .$$

Comparing these formulas with the expression for $\Delta(\rho)$ we see that

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)} = \left[\frac{\Omega_{b\nu} \cdot \delta_{b\nu}}{\Omega_b \cdot \delta_b}\right] \sigma^{-\lambda_p} e^{-\sigma w_\nu} \ .$$

2.) $p < \nu \leq n-m$. Here we reason as under 1.), the only difference being that $p' = p$ and therefore

$$\sum_{\omega=1}^{p'} w'_\omega = \sum_{\omega=1}^{p} w_\omega$$

and

$$T_{b\nu} = \sum_{\omega=1}^{p} \lambda_\omega + \sum_{\mu=r+1}^{r+n-m-p-1} \tau_\mu \ .$$

Hence, in this case

$$\frac{\Delta_\nu(\rho)}{\Delta\ (\rho)} = \left[\frac{Q_{b\nu} \cdot \delta_{\nu\nu}}{Q_b \cdot \sigma_b}\right] \sigma^{-\tau_{r+n-m-p}} .$$

Q.E.D.

19. Now we are prepared to prove a theorem, which, of course, is a part of the statement of the Main Theorem.

<u>Theorem 2</u>: If the assumptins 1° - 9° of section 3 are satisfied, if, furthermore,

$$0 < r-p < m$$

and

$$n-m \not\equiv 0 \pmod 4, \quad \text{and} \quad b_0 < 0$$

or

$$n-m \not\equiv 2 \pmod 4, \quad \text{and} \quad b_0 > 0 ,$$

then, as $\rho \to \infty$, the solution of the problem defined by (101) and (108) converges in $\alpha < x < \beta$ to that solution of the "limiting" differential equation

$$M(y) = 0$$

which satisfies the m boundary conditions not canceled in application of the rule of the Main Theorem.

<u>Proof</u>: We substitute in (133) the values of $\frac{\Delta_\nu(\rho)}{\Delta\ (\rho)}$ obtained in sections 17 and 18 and replace $U_\nu(x,\rho)$ by its values as given by (114) and (115). Then we find

$$\begin{aligned} U(x,\rho) = & \sum_{\nu=1}^{p} \sigma^{-\lambda_p} e^{\sigma\{\int_\alpha^x \varphi_\nu(\xi)d\xi - w_\nu\}} [\pi_\nu \cdot \eta(x)] \\ & + \sum_{\nu=p+1}^{n-m} \sigma^{-\tau_{r+n-m-p}} e^{\sigma \int_\alpha^x \varphi_\nu(\xi)d\xi} [\pi_\nu \cdot \eta(x)] \qquad (169) \\ & + \sum_{\mu=1}^{m} [\frac{\delta_{b,n-m+\mu}}{\delta_b}] u_\mu(x) . \end{aligned}$$

It is easy to see that the first two sums of the right member of (169) tend to zero, in $\alpha < x < \beta$, as ρ, and therefore σ, tend to infinity. For,

$$\mathrm{Re}\left(\int_\alpha^x \varphi_\nu(\xi)d\xi - w_\nu\right) < 0, \quad \text{in} \quad \alpha \leq x < \beta, \quad \text{for} \quad \nu \leq p$$

and

$$\mathrm{Re}\left(\int_\alpha^x \varphi_\nu(\xi)d\xi\right) < 0, \quad \text{in} \quad \alpha < x \leq \beta, \quad \text{for} \quad \nu > p \quad ,$$

in consequence of (117), (128) and the definition of p.

Therefore

$$u(x) = \lim_{\rho\to+\infty} U(x,\rho) = \sum_{\mu=1}^{m} \frac{\delta_{b,n-m+\mu}}{\delta_b} u_\mu(x) \quad . \tag{170}$$

But if we remember the definition of δ_b, $\delta_{b,n-m+\mu}$ and $u_\mu(x)$, as given in (159), in section 17, and in theorem 1 (section 6), respectively, then we see that the right member of (170) is just the solution of $M(y) = 0$ satisfying the boundary conditions not canceled in application of the Main Theorem, and condition (160) is seen to be equivalent with assumption 8° of the Main Theorem in the convergent cases I - III (table (113)).

Q.E.D.

20. <u>Remark</u>: Formula (169) is, in fact, a complete asymptotic solution of the boundary value problem. It might be used for a more detailed description of the boundary layer phenomenon. It can be written in the following simpler and more symmetric form:

$$\begin{aligned} U(x,\rho) = u(x) &+ \sigma^{-\tau} \sum_{\nu=1}^{p} e^{\sigma\int_\alpha^x \varphi_\nu(\xi)d\xi} [\pi_\nu \eta(x)] \\ &+ \sigma^{-\lambda} \sum_{\nu=p+1}^{n-m} e^{\sigma\int_\alpha^x \varphi_\nu(\xi)d\xi} [\pi_\nu \eta(x)] \end{aligned} \tag{171}$$

where $\tau = \tau_{r+n-m-p}$ and $\lambda = \lambda_p$ are the lowest orders of differentiation occurring in the canceled boundary conditions on each side. An easy consequence of (169) is, e.g., the following interesting

<u>Corollary</u>: Under the assumptions of theorem 2 the derivatives

$$U(\alpha,\rho),\ U'(\alpha,\rho),\ldots,U(\alpha,\rho)^{(\tau_{r+n-m-p}-1)}$$

converges to the value of the corresponding derivatives of $u(x)$ at $x = \alpha$. The next derivative,

$$U^{(\tau_{r+n-m-p})}(\alpha,\rho)$$

converges, as $\rho \to \infty$, but in general the limit is not equal to $u^{(\tau_{r+n-m-p})}(\alpha)$. All the higher derivatives of $U(x,\rho)$ tend to infinity at $x = \alpha$. An analogous statement can be made at $x = \beta$, with $U^{(\lambda_p)}(\beta,\rho)$ as the last convergent derivative.

In less precise language we may express the statement of this corollary by saying that the last canceled boundary condition at each end point determines the derivative of $U(x,\rho)$ in which the boundary layer occurs at that endpoint.

§7. <u>The Proof of the Divergence in the Cases IA and IC</u>.

21. We are now going to show that in the two remaining cases $U(x,\rho)$ tends to infinity as $\rho \to \infty$. To that end it is sufficient to prove that <u>one</u> of the terms in the right member of (133) tends to infinity. For each of the first $n-m$ terms of (133) is of the form

$$K_\nu \sigma^{\gamma_\nu} e^{\sigma W_\nu(x)}$$

If one of these terms, say $K_1 \sigma^{\gamma_1} e^{\sigma W_1(x)}$ tends to infinity, the whole sum can remain bounded only if the sum of all terms of the same order of magnitude as this particular term vanishes identically. This would require that at least one other term has the same exponential factor as K_1. We would have, e.g.,

$$W_1(x) = W_2(x) \quad .$$

Now all the $W_\nu(x)$ are of the form

$$W_\nu(x) = V_\nu + \int_\alpha^x \varphi_\nu(\xi)d\xi$$

where the constant V_ν originates from the factor $\frac{\Delta_\nu(\rho)}{\Delta(\rho)}$ while $\int_\alpha^x \varphi_\nu(\xi)d\xi$ is the contribution of $U_\nu(x,\rho)$. But an equation like

$$V_1 + \int_\alpha^x \varphi_1(\xi)d\xi = V_2 + \int_\alpha^x \varphi_2(\xi)d\xi$$

is impossible, since no two $\varphi_\nu(x)$ are equal.

22. Proof of the divergence in the case IA:

Let us calculate the determinant $\Delta_{r+1}(\rho)$. In this case the formulas for $\Delta(\rho)$ can be applied, if we replace

$$p \text{ by } p' = p-1 \quad (\text{since } r+1 < p) \quad ,$$

$$m \text{ by } m' = m+1 \quad ,$$

and the sequence

$$w_1, w_2, \ldots, w_{n-m}$$

by the sequence

$$w_1', w_2', \ldots, w_{n-m-1}'$$

obtained by omitting the term w_{r+1} from the sequence of the w_ν. As we have

$$r-p' < m'$$

the formula to be used is (165). We obtain

$$\Delta_{r+1}(\rho) = [\Omega_{a,r+1} \cdot \delta_{a,r+1}]\,\sigma^{T_{a,r+1}}\, e^{\sigma \sum_{\omega=1}^{r} w_\omega'} \quad ,$$

where

$$T_{a,r+1} = \sum_{\omega=1}^{r} \lambda_\omega + \sum_{\mu=r+1}^{n-m'} \tau_\mu \ .$$

It follows that

$$\sum_{\omega=1}^{r} w'_\omega = \sum_{\omega=1}^{r} w_\omega$$

and

$$T_{a,r+1} = \sum_{\omega=1}^{r} \lambda_\omega + \sum_{\mu=r+1}^{n-m-1} \tau_\mu \ .$$

For $\dfrac{\Delta_{r+1}(\rho)}{\Delta(\rho)}$ we obtain

$$\frac{\Delta_{r+1}(\rho)}{\Delta\ (\rho)} = \left[\frac{\Omega_{a,r+1}}{\Omega_a} \frac{\delta_{a,r+1}}{\delta_a}\right] \sigma^{-\tau_{n-m}}$$

and the $r+1$ st term of (133) becomes therefore

$$\frac{\Delta_{r+1}(\rho)}{\Delta\ (\rho)} U_{r+1}(x,\rho) = \left[\frac{\Omega_{a,r+1}}{\Omega_a} \cdot \frac{\delta_{a,r+1}}{\delta_a} \cdot \eta(x)\right] \sigma^{-\tau_{n-m}} e^{\sigma\int_\alpha^x r_{r+1}(\xi)\,d\xi} \ .$$

Since $r+1 \leq p$, the exponential factor of the right member tends to infinity as $\rho \to \infty$. If we can prove that the expression in brackets does not vanish, then the divergence of $U(x,\rho)$ is assured. $\eta(x)$ does not vanish in $\alpha \leq x \leq \beta$ (see (118)). For $\Omega_{a,r+1}$ we can prove that the following lemma is true.

Lemma 4: $\Omega_{a,r+1} \neq 0$, if assumption 9°' of the Main Theorem is satisfied.

Proof: An almost literal repetition of the arguments of §5, applied to $\Delta_{r+1}(\rho)$ instead of to $\Delta(\rho)$, which we shall omit, shows that $\Omega_{a,r+1}$ is different from zero, if the following determinants do not vanish:

$$A_{a,r+1} = \begin{vmatrix} \varphi_1^{\lambda_1}(\beta) & \cdots & \varphi_r^{\lambda_1}(\beta) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \varphi_1^{\lambda_r}(\beta) & \cdots & \varphi_r^{\lambda_r}(\beta) \end{vmatrix}$$

and

$$B_{a,r+1} = \begin{vmatrix} \varphi_{r+2}^{\tau_{r+1}}(\alpha) & \cdots & \varphi_{n-m}^{\tau_{r+1}}(\alpha) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \varphi_{r+2}^{\tau_{n-m-1}}(\alpha) & \cdots & \varphi_{n-m}^{\tau_{n-m-1}}(\alpha) \end{vmatrix} .$$

To these determinants the reasoning of lemma 2b can be applied, leading exactly to lemma 4.

$\delta_{a,r+1}$ is a determinant one column of which is formed of $m+1$ of the n numbers ℓ_i. $\delta_{a,r+1}$ will be zero for certain exceptional vlues of these ℓ_i. But even the assumption that the ℓ_i have these exceptional values would not be sufficient to guarantee the convergence of $U(x,\rho)$, for the $r+1$ st term of (133) will, in general, not be the only one that has an exponential factor tending to infinity. We shall omit the not difficult problem of finding sufficient conditions for the convergence. We will assume instead, without mentioning it each time, that the ℓ_i do not have these exceptional values.

From the preceding considerations it follows that $U(x,\rho)$ is in fact divergent in case IA, provided that the assumptions of the Main Theorem are satisfied.

23. The proof for the divergence of $U(x,\rho)$ in the case IC is analogous to the proof for the case IA, if the $(r-m)$-th term of the sum in (133) is considered, instead of the $(r+1)$st, on which the proof in the case IA was based. One obtains the expression

$$\frac{\Delta_{r-m}(\rho)}{\Delta(\rho)} = \left[\frac{Q_{c,r-m}\cdot\delta_{c,r-m}\cdot\eta(x)}{Q_c\cdot\delta_c}\right]\sigma^{-\lambda_{r-m}}e^{\sigma\{\int_\alpha^x\varphi_{r-m}(\xi)d\xi - w_{r-m}\}}$$

from which the divergence of $U(x,\rho)$ follows as in the case IA, since $Re(\varphi_{r-m}) < 0$.

§8. The Case II.

24. The case II, i.e. the case when $Re(\varphi_{p+1}) = Re(\varphi_{p+2}) = 0$, requires a special discussion, because in our reasoning in case I we assumed repeatedly - notably in section 14 - that every $L_i(U_\nu)$, with $i < r$, is either of greater or of lower order of magnitude than all $L_i(u_\nu)$ with $i > r$. But this is no longer true in case II, for the $L_1(U_{p+1})$ and $L_1(U_{p+2})$, because the exponentials $e^{\sigma w_{p+1}}$ and $e^{\sigma w_{p+2}}$ have always the absolute value 1. We must therefore modify our considerations for case II, from section 13 onward.

We remind the reader of the remark made in section 13, to the effect that case II of (139) is equivalent with case IV of (113).

25. The case IIA.

Re-reading sections 14-22 one sees that no modification of the proofs for case IA is necessary to obtain the proofs for case IIA. All arguments remain literally the same. The reason is that, for $r < p$, $e^{\sigma w_{p+1}}$ and $e^{\sigma w_{p+2}}$ do not appear at all in the asymptotic expressions for $\Delta(\rho)$ or $\Delta_{r+1}(\rho)$.

26. The case IIB.

This is no longer true for case IIB, defined by the inequalities $0 < r-p < m+2$. In order to find the minor of maximal order of magnitude among the minors of the first $n-m$ columns of $\Delta(\rho)$ we go back to lemma 1, section 12. Whether the number h in (136) is chosen equal to $p, p+1, p+2$, does not have any influence on the real part of $\sum_{\nu=1}^{h} w_\nu$, but any other value of h leads to a $\sum_{\nu=1}^{h} w_\nu$ with a smaller real part. Which of these three values has to be chosen for h in order to make the order of (136) a maximum depends therefore on (137). h must be that one of the three numbers $p, p+1, p+2$ for which the

λ_{i_ν} and the τ_{i_μ} can be chosen so as to make T a maximum. Let us call this maximum $T_b^{(II)}$. As in section 14 it is clear that $\lambda_1, \lambda_2, \ldots, \lambda_p$ must belong to $T_b^{(II)}$, as well as $\tau_{r+1}, \tau_{r+2}, \ldots, \tau_{r+n-m-p-2}$.

With respect to the two remaining terms to be chosen we can only say that they must be the two largest of the remaining λ_i's and τ_i's. This proves that the *minor of greatest* order of magnitude among all minors of the first $n-m$ columns of $\Delta(\rho)$ is in this case formed by the rows

$$1,2,\ldots,p;\ r+1,r+2,\ldots,r+n-m-p-2\ , \tag{172}$$

and two more rows, which must be those corresponding to the boundary conditions containing the highest order of differentiation excluding those rows already contained in the sequence (172). A comparison with part B of the third step of the Main Theorem shows that these are just the *rows of* $\Delta(\rho)$ belonging to boundary conditions that must be canceled in application of that rule. We know from the proof in case I why this is so: the rows of $\Delta(\rho)$ appearing in the minor of maximal order ot the first $n-m$ columns of $\Delta(\rho)$ are just those that do not occur in the cofactor of this minor, and this *cofactor* determined the boundary conditions satisfied by the limit of $U(x,\rho)$, if it exists.

The exceptional case of indetermination occus, when the two additional rows after the rows (172) have been chosen, are not uniquely determined. We shall assume, for the present, that we have to do with the regular case. The case of indetermination will be treated in §9.

Our rule is so formulated that it takes also care of the case that λ_{p+1} or λ_{p+2} do not exist because $r = p$ or $r = p+1$.

27. The minor $D_b^{(II)}(\rho)$ of maximal order or magnitude among the minors of the first $n-m$ columns of $\Delta(\rho)$ can now be written

$$D_b^{(II)}(\rho) = [Q_b^{(II)}]\ \sigma^{T_b^{(II)}}\ e^{\sigma \sum_{\nu=1}^{H} w_\nu} \tag{173}$$

where

$$T_b^{(II)} = \sum_{\nu=1}^{H} \lambda_\nu + \sum_{\mu=r+1}^{r+n-m-H} \tau_\mu , \tag{174}$$

and H is equal to p, $p+1$, or $p+2$, as the case may be.

As in case I we can then prove that $Q_b^{(II)} \neq 0$, provided assumption 9° is satisfied The cofactor of $D_b^{(II)}(\rho)$ will again be of the form

$$[\delta_b^{(II)}]$$

where $\delta_b^{(II)}$ is defined in analogy with δ_b. Assumption 8° of the Main Theorem assures us again that

$$\delta_b^{(II)} \neq 0 \quad .$$

Finally, we find for $\Delta(\rho)$, similar to (161)

$$\Delta(\rho) = [Q_b^{(II)} \delta_b^{(II)}] \, \sigma^{T_b^{(II)}} \, e^{\sigma \sum_{\nu=1}^{H} w_\nu} \quad \text{in case IIB} \quad . \tag{175}$$

28. As in case I, we could now calculate the determinants $\Delta_\nu(\rho)$ by the method used for $\Delta(\rho)$. But since we are only interested in proving that

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)} U_\nu(x,\rho)$$

tends to zero, for $\nu < n-m$, we shall not calculate the analogue of (169) for this case. As in proof of lemma 3 we consider first the $\Delta_\nu(\rho)$ for $\nu < p$. For these ν the determinant $\Delta_\nu(\rho)$ does not contain the column with $e^{\sigma w_\nu}$ of $\Delta_\nu(\rho)$. It follows from lemma 1, that then the minor of maximal order among the minors of the first $n-m-1$ columns of $\Delta_\nu(\rho)$ cannot contain as factor an exponential of higher order than

$$e^{\sigma \sum_{\omega=1}^{H} w_\omega - w_\nu} .$$

This is therefore also the exponential of maximal order that can occur in the asymptotic expression for $\Delta_\nu(x)$. Hence,

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)}$$

has the exponential factor $e^{-\sigma w_\nu}$ (and possibly exponential factors of order 1), and

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)} U_\nu(x,\rho) \quad , \quad (\nu < p)$$

has an exponential factor whose exponent is

$$\sigma\left(\int_\alpha^x \varphi_\nu(\xi)d\xi - w_\nu\right)$$

an expression which tends to zero as $\sigma \to \infty$.

If $\nu > p+2$, we prove similarly that $\frac{\Delta_\nu(\rho)}{\Delta(\rho)}$ does not contain any exponential factors (except possibly exponentials of the order or magnitude of 1). Hence the asymptotic expression for

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)} U_\nu(x,\rho) \quad , \quad (\nu > p+2)$$

contains an exponential factor whose exponent is

$$\sigma \int_\alpha^x \varphi_\nu(\xi)d\xi \quad , \quad (\nu > p+2) \quad ,$$

and this expression tends to zero as $\sigma \to \infty$.

29. The two remaining terms of (133) require a more careful analysis. We treat only the case $\nu = p+1$, the case $\nu = p+2$ being almost identical. Similar as in section 26 we ask which choice of $n-m-1$ rows of $\Delta_{p+1}(\rho)$ leads to the minor of highest order among the minors of the first $n-m-1$ columns of $\Delta_{p+1}(\rho)$. This minor will have either

$$e^{\sigma \sum_{\omega=1}^{p} w_\omega} \quad \text{or} \quad e^{\sigma \sum_{\omega=1}^{p} w_\omega + w_{p+2}}$$

as exponential factor. In both cases the asymptotic expression for

$$\frac{\Delta_{p+1}(\rho)}{\Delta(\rho)} U_{p+1}(x,\rho)$$

will contain an exponential factor of the order of 1. But the asymptotic expression for $\Delta_{p+1}(\rho)$ has also a factor of the form $\sigma^{T'}$, and similarly as in section 26 we conclude that T' is a sum which contains the terms $\lambda_1, \lambda_2, \ldots, \lambda_p$ and the terms τ_{r+1}, $\tau_{r+2}, \ldots, \tau_{r+n-m-p-2}$. To these terms one more term has to be added (not two terms as in the case of $\Delta(\rho)$, because $\Delta_{p+1}(\rho)$ must be expanded in terms of its first $n-m-1$, not $n-m$, columns). This term must be the largest of the remaining λ_i's and τ_i's. Finally, in the asymptotic calculation of $\frac{\Delta_{p+1}(\rho)}{\Delta(\rho)}$, we have to form $\sigma^{T'-T_b^{(II)}}$, and this will be the power of σ occurring in the asymptotic expression for $\frac{\Delta_{p+1}(\rho)}{\Delta(\rho)} U_{p+1}(x,\rho)$. This proves that the order of magnitude of $\frac{\Delta_{p+1}(\rho)}{\Delta(\rho)} U_{p+1}(x,\rho)$ is not greater than that of

$$\sigma^{-s}$$

where s is the <u>smaller</u> of the two terms chosen for $T_b^{(II)}$, after $\lambda_1, \lambda_2, \ldots, \lambda_p$ and $\tau_{r+1}, \tau_{r+2}, \ldots, \tau_{r+n-m-p-2}$ have been selected. s may be any of the numbers

$$\lambda_{p+1}, \lambda_{p+2}, \tau_{r+n-m-p-1}, \tau_{r+n-m-p} \quad . \tag{176}$$

In order to prove that the $p+1$ st term of (133) tends to zero it remains only to show that none of the numbers (176) can be zero:

a) $s = \lambda_{p+1}$. In this case the other one of the two last numbers chosen for $\tau_b^{(II)}$ must be $\tau_{r+n-m-p-1}$, because, if it were λ_{p+2}, s would not be the smaller of the two, and it cannot be $\tau_{r+n-m-p}$ since

$$\tau_{r+n-m-p} < \tau_{r+n-m-p-1}$$

and the two chosen numbers must be the largest of the numbers (176). Since s must be the smaller of these two numbers, we conclude that

$$\lambda_{p+1} < \tau_{r+n-m-p-1} \quad .$$

On the other hand,

$$\lambda_{p+1} > \tau_{r+n-m-p} \quad . \tag{177}$$

If $\lambda_{p+1} = 0$. λ_{p+1} must be the last of the numbers λ_i hence

$$p+1 = r \quad . \tag{178}$$

It then follows from (177) that $\tau_{r+n-m-p}$ cannot exist, in other words

$$\tau_{r+n-m-p-1} = \tau_n$$

or

$$r+n-m-p-1 = n \quad . \tag{179}$$

(178) and (179) imply

$$m = 0 \quad ,$$

a case excluded from our considerations. Hence, $\lambda_{p+1} > 0$.

b) $s = \lambda_{p+2}$. Similarly as in a) it follows that

$$\lambda_{p+2} > \tau_{r+n-m-p-1} \quad .$$

$\lambda_{p+2} = 0$ would imply $p+2 = r$ and $r+n-m-p-2 = n$, hence

$$m = 0 \quad .$$

c) and d) $s = \tau_{r+n-m-p-1}$ or $s = \tau_{r+n-m-p}$. We leave the proofs in these cases to the reader, since they follow exactly the pattern of the proofs in a) and b) with the same result.

30. Thus we have finished the proof of the following theorem.

Theorem 3: If the assumptions 1° - 9° of section 3 are satisfied, if

$$0 < r-p < m+2 \quad ,$$

and if

$$n-m \equiv 0 \pmod 4, \quad \text{and} \quad b_0 < 0$$

or

$$n-m \equiv 2 \pmod 4, \quad \text{and} \quad b_0 > 0 \quad ,$$

then, as $\rho \to \infty$, the solution $U(x,\rho)$ of the problem defined by (101) and (108) converges in $\alpha < x < \beta$ to that solution of the limiting differential equation $M(y) = 0$ which satisfies the m boundary conditions not canceled in application of the rule of the Main Theorem for the case IV, unless we have to do with the case of indetermination.

31. The case IIC. In case IIC (i.e., if $r-p > m+2$) one can again, as in section 23, consider the $(r-m)$th term of the right member of (133) and prove that its asymptotic expression contains the exponential factor

$$e^{\sigma\{\int_\alpha^x \varphi_{r-m}(\xi)d\xi - w_{r-m}\}} \quad .$$

This is sufficient to show that this term must tend to infinity, since this exponential tends to infinity *more strongly than any power of* σ may tend to zero.

§9. The Case of Indetermination.

32. Let us assume that we cannot decide in a uniquely determined way, which is the last row of $\Delta(\rho)$ to be chosen in order to obtain the minor of greatest order of magnitude among the minors of the first $n-m$ columns of $\Delta(\rho)$. This is the case which we have left aside in section 26. We have seen there that this occurrence means that we have to do with the case of indetermination of the Main Theorem.

In this case we must have

$$\lambda_\ell = \tau_k$$

where ℓ is one of the numbers $p+1$, $p+2$ and k one of the numbers $r+n-m-p-1$, $r+n-m-p$.

Furthermore, λ_ℓ and τ_k must be the second and third in size of the numbers λ_i and τ_i, excluding the numbers $\lambda_1, \lambda_2, \ldots, \lambda_p$ and $\tau_{r+1}, \tau_{r+2}, \ldots, \tau_{r+n-m-p-2}$. For simplicity let us assume that $\ell = p+1$. (The reasoning is the same for $\ell = p+2$.) Denote by $D_1(\rho)$ and $D_2(\rho)$ the minors of the first $n-m$ columns of the determinant $\Delta(\rho)$, with the rows

$$1, 2, \ldots, p,\ p+1;\ r+1,\ r+2, \ldots, k-1$$

$$1, 2, \ldots, p; \qquad r+1,\ r+2, \ldots, k-1,\ k\ ,$$

respectively. Then $D_1(\rho)$ and $D_2(\rho)$ are both minors of the same maximal order, and we cannot reason as in section 16. But, using formula (138) we can write

$$D_1(\rho) = F(\rho)\ e^{\sigma w_{p+1}}\ [d_1]$$

$$D_2(\rho) = F(\rho) \quad \cdot \quad [d_2]\ ,$$

where

$$F(\rho) = \sigma^{\sum_{\nu=1}^{p} \lambda_\nu + \sum_{\mu=1}^{k} \tau_\mu}\ e^{\sigma \sum_{\nu=1}^{p} w_\nu}$$

d_1 and d_2 are certain constants, which can easily be proved to be different from zero by the method of section 15, provided assumption 9° is satisfied for each of the two ways of canceling corresponding to $D_1(\rho)$ and to $D_2(\rho)$, respectively.

The cofactors of the minors $D_1(\rho)$ and $D_2(\rho)$ in $\Delta(\rho)$ can be written in the form $[\delta_1]$ and $[\delta_2]$, where δ_1 and δ_2 are certain constants. If assumption 8° is satisfied for both ways of applying the cancellation rule of the Main Theorem, then we can be sure that

$$\delta_1 \neq 0, \ \delta_2 \neq 0 \ .$$

Expanding $\Delta(\rho)$ in terms of its first $n-m$ columns we find then

$$\Delta(\rho) = D_1(\rho)[\delta_1] \pm D_2(\rho)[\delta_2] = F(\rho)\{e^{\sigma w_{p+1}}[d_1\delta_1] \pm [d_2\delta_2]\} \ . \qquad (180)$$

33. While in all cases so far the last m terms of the sum (133) were convergent, so that the convergence or non-convergence of $U(x,\rho)$ depended entirely upon the first $n-m$ terms of (133), in this case the last m terms of (133) will, in general, be divergent.

In fact, let $n-m < j \leq n$. Then we find immediately in analogy with (180),

$$\Delta_j(\rho) = F(\rho)\{e^{\sigma w_{p+1}}[d_1\delta_{1j}] + [d_2\delta_{2j}]\} \quad , \tag{181}$$

where $[\delta_{1j}]$ and $[\delta_{2j}]$ are the determinants obtained instead of $[\delta_1]$ and $\delta_2]$, if the j-th column of $\Delta(\rho)$ is replaced by

$$\begin{matrix} \ell_1 \\ \ell_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ell_n \end{matrix} \quad .$$

(180), (181) and (115) show that the j-th (j n-m) term of (133) is equal to

$$\frac{\Delta_j(\rho)}{\Delta(\rho)} U_j(x,\rho) = \frac{e^{\sigma w_{p+1}}[d_1\delta_{1j}] + [d_2\delta_{2j}]}{e^{\sigma w_{p+1}}[d_1\delta_1] + [d_2\delta_2]} [u_{j-n-m}(x)] \tag{182}$$

and that

$$\sum_{j=n-m}^{n} \frac{\Delta_j(\rho)}{\Delta(\rho)} U_j(x,\rho) = \frac{e^{\sigma w_{p+1}} \sum\limits_{j=n-m}^{n} [d_1\delta_{1j} u_{j-n+m}(x)] + \sum\limits_{j=n+m}^{n} [d_2\delta_{2j} u_{j=n-m}(x)]}{e^{\sigma w_{p+1}} [d_1\delta_1] + [d_2\delta_2]} \quad . \tag{183}$$

Because of the oscillating factor $e^{\sigma w_{p+1}}$ this expression can converge only if the determinant

$$\begin{vmatrix} \sum\limits_{j=n-m}^{n} d_1\delta_{1j}u_{j=n-m}(x) & \sum\limits_{j=n-m}^{n} d_2\delta_{2j}u_{j=n+m}(x) \\ d_1\delta_1 & d_2\delta_2 \end{vmatrix}$$

vanishes, i.e. if

$$\sum_{j=n-m}^{n} \begin{vmatrix} \delta_{1j} & \delta_{2j} \\ \delta_1 & \delta_2 \end{vmatrix} u_{j-n+m}(x) = 0 \quad .$$

Since the $u_\mu(x)$, $(\mu = 1,2,\ldots,m)$ are, by assumption, linearly independent, this leads to the m conditions

$$\delta_2\delta_{1j} - \delta_1\delta_{2j} = 0 \quad , \quad (j = n-m+1,\ldots,n) \quad . \tag{184}$$

The left sides of (184) depend on the numbers $\ell_1, \ell_2,\ldots,\ell_n$. (184) represents therefore a set of conditions on the prescribed boundary values. As in the previous cases we assume that the ℓ_i's do not have the very special values required by (184). Then, the value of (183) oscillates, as $\sigma \to \infty$, but remains bounded.

As to the first $n-m$ terms of the sum in (133), it can be proved exactly as in the regular case IIB (section 27-29) that they all tend to zero.

This completes the proof of the non-convergence of $U(x,\rho)$ in the case of indetermination and also the proof of the whole Main Theorem.

34. <u>Remark</u>: Going over our whole proof we see that the assumption $m > 0$ was not used at all in the proof for case I, i.e., for the cases (I) - (III) of (113). Hence, all our results in these cases remain valid for $m = 0$ also.

In case II, i.e. (IV), the hypothesis $m > 0$ was used only to exclude a rather special occurrence in section 29. It would not be difficult to formulate a general theorem for the case $m = 0$ also.

Chapter II

FURTHER RESULTS IN THE CASE $n-m = 1$

§1. Generalization of the Boundary Conditions.

35. In the special case $n-m = 1$ it is not difficult to replace the boundary conditions (108) by the more general ones

$$L_i(y) = \sum_{\nu=1}^{n} \alpha_{i\nu} y^{(\nu-1)}(\alpha) + \sum_{\mu=1}^{n} \beta_{i\mu} y^{(\mu-1)}(\beta) = \ell_i, \quad (i = 1,2,\ldots,n) \quad , \tag{201}$$

where the constants $\alpha_{i\nu}$, $\beta_{i\mu}$ and ℓ_i are only restricted by the condition that the n boundary conditions are independent and compatible.

We note first: If, in (201), the $L_i(y)$ and the ℓ_i are subjected to the same linear transformation with constant coefficients and non-vanishing determinant, then the resulting equations

$$L_i'(y) = \sum_{\nu=1}^{n} \alpha_{i\nu}' y^{(\nu-1)}(\alpha) + \sum_{\mu=1}^{n} \beta_{i\mu}' y^{(\mu-1)}(\beta) = \ell_i' \tag{202}$$

constitute a set of boundary conditions equivalent to (201) in the sense that a function satisfying (201) satisfies also (202) and vice versa.

From this remark we see that we can assume without loss of generality that not all the $\alpha_{i\nu}$ or all the $\beta_{i\mu}$ are zero, because in that case the boundary conditions (201) would be equivalent to ordinary initial conditions, which are a special case of boundary conditions of the type (108).

36. Denote by s the greatest value of ν for which at least one of the $\alpha_{i\nu}$ is not zero, and by t the greatest value of μ for which at least one of the $\beta_{i\mu}$ is not zero. There exists a non-degenerate linear transformation with constant coefficients transforming (201) into the system of equivalent boundary conditions

$$ {}_{\alpha}L_i \equiv \sum_{\nu=1}^{n} {}_{\alpha}a_{i\nu}\, y^{(\nu-1)}(\alpha) + \sum_{\mu=1}^{n} {}_{\alpha}\beta_{i\mu}\, y^{(\mu-1)}(\beta) = {}_{\alpha}\ell_i \qquad (203) $$

having the property that

$$ {}_{\alpha}\alpha_{1t} \neq 0, \quad {}_{\alpha}\alpha_{2t} = \cdots\cdots = {}_{\alpha}\alpha_{ns} = 0 \quad . $$

Such a transformation can be chosen in many ways. Similarly, there is a transformation changing (201) into

$$ {}_{\beta}L_i \equiv \sum_{\nu=1}^{n} {}_{\beta}\alpha_{i\nu}\, y^{(\nu-1)}(\alpha) + \sum_{\mu=1}^{n} {}_{\beta}\beta_{i\mu}\, y^{(\mu-1)}(\beta) = {}_{\beta}\ell_i \qquad (204) $$

having the property that

$$ {}_{\beta}\beta_{1t} \neq 0, \quad {}_{\beta}\beta_{2t} = \cdots\cdots = {}_{\beta}\beta_{nt} = 0 \quad . $$

For $n-m = 1$ equation (114) and (115) reduce to

$$ U_1(x,\rho) = e^{-\rho\int_{\alpha}^{x} b_0(\xi)d\xi}\,[\eta(x)] $$

$$ U_{1+\mu}(x,\rho) = [u_{\mu}(x)] \quad , \quad (\mu = 1,2,\ldots,n-1) \quad . $$

Let us assume first that $b_0(x) > 0$. Then we use (203) instead of (201) and see immediately that

$$ {}_{\alpha}L_1(U_1) = \pm\, \rho^{s-1}[b_0(\alpha)^{s-1} \cdot \eta(\alpha)] $$

is of greater order of magnitude than all the other $L_i(U_i)$, $(i = 2,3,\ldots,n)$. Solving equations (125) asymptotically in this case, by the method used in the first chapter, we see that we can reason exactly as there. Of the assumptions of the Main Theorem we need only 1°, 2°, 3°, 5°, 6° and an assumption corresponding to 8° which states that

$$\delta_\alpha = \begin{vmatrix} {}_\alpha L_2(u_1), & \cdots, & {}_\alpha L_2(u_n) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ {}_\alpha L_n(u_1), & \cdots, & {}_\alpha L_n(u_n) \end{vmatrix} \neq 0 \qquad (205)$$

We see then easily that the solution of (101) and (201) tends, for $n-m = 1$, and $b_0 > 0$, to that solution of $M(y) = 0$ which satisfies the boundary conditions

$$ {}_\alpha L_i(y) = {}_\alpha \ell_i \quad , \quad (i = 2,3,\ldots,n) \qquad (206)$$

but not the boundary condition ${}_\alpha L_1(y) = {}_\alpha \ell_1$ except for special values of the ℓ_i's. Using a similar reasoning in the case $b_0(x) < 0$ we find that in that case $u(x) = \lim_{\rho\to\infty} U(x,\rho)$ satisfies $M(y) = 0$ and the boundary conditions

$$ {}_\beta L_i(y) = {}_\beta \ell_i \quad , \quad (i = 2,3,\ldots,n) \qquad (207)$$

provided

$$\delta_\beta = \begin{vmatrix} {}_\beta L_2(u_1), & \cdots, & {}_\beta L_2(u_n) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ {}_\beta L_n(u_1), & \cdots, & {}_\beta L_n(u_n) \end{vmatrix} \neq 0 \qquad (208)$$

This result can be formulated in a somewhat more symmetrical form. To that end note that the boundary conditions (206) do not involve any more the highest derivative at $x = \alpha$ occurring in (201). Any linear combination of the equations (201) which does not contain $y^{(s-1)}(\alpha)$ must be linearly dependent on equations (206) and no linear combination of equations (206) contains $y^{(s-1)}(\alpha)$. Similarly for (207) with respect to $y^{(t-1)}(\beta)$. Hence, we can state the following theorem.

Theorem 4: If $n-m = 1$ and if the conditions 1°, 2°, 3°, 5°, 6° of the Main Theorem of chapter I as well as (205) and (208) are satisfied, then the function $U(x,\rho)$ satisfying (101) and the boundary conditions (201) tends with increasing ρ to a solution $u(x)$ of $M(y) = 0$. According as $b_0 > 0$ or $b_0 < 0$ the function $u(x)$ satisfies all boundary conditions that depend linearly on (201) and do not contain the highest derivative at $x = \alpha$ or $x = \beta$, respectively, occurring in (201).

Remark: The conditions (205) and (208) can be formulated in a way independent of the particular choice of the fundamental system $u_\mu(x)$, $(\mu = 1,2,\ldots,n-1)$, by saying that we assume that only the function $u(x) \equiv 0$ satisfies the differential equation $M(y) = 0$ and the homogeneous boundary conditions corresponding to (206) or (207), respectively.

Example: $n = 3$, $m = 2$,

$$\left.\begin{aligned} y(\alpha) - y'(\alpha) + y''(\beta) &= \ell_1 \\ y(\alpha) + y'(\alpha) + y(\beta) - y'(\beta) &= \ell_2 \\ y'(\alpha) - 2y''(\beta) &= \ell_3 \end{aligned}\right\} .$$

If $b_0 > 0$, then $\lim\limits_{\rho\to\infty} U(x,\rho)$ satisfies the boundary conditions

$$2y(\alpha) + y(\beta) - y'(\beta) + y''(\beta) = \ell_1 + \ell_2$$

$$y(\alpha) + y(\beta) - y'(\beta) + 2y''(\beta) = \ell_2 - \ell_3 .$$

But if $b_0 < 0$, then $\lim\limits_{\rho\to\infty} U(x,\rho)$ satisfies the boundary conditions

$$y(\alpha) + y'(\alpha) + y(\beta) - y'(\beta) = \ell_2$$

$$2y(\alpha) - y'(\alpha) = 2\ell_1 + \ell_3 .$$

§2. The "Stretching" of the Boundary Layer

37. For the relatively simple types of boundary layer problems with which this investigation is concerned we have been able to develop a method that allows us to calculate asymptotic expressions for the solution of the boundary value problem (compare, e.g., formula (171)). From these asymptotic expressions one can easily obtain all desired information about the behavior of the solution of the boundary value problem near the endpoints for large values of ρ. (Compare, e.g., the corollary in section 20).

In the more complicated boundary layer problems occurring in physics such complete asymptotic solutions are often not available. In those cases it is customary to transform the given boundary value problem, by a change of the independent variable, into a new boundary value problem which does not tend to a problem of lower order when ρ tends to infinity.

As an example for such a transformation we take the differential equation (101), for the special case

$$b_0(x) > 0 \quad .$$

The case $b_0(x) < 0$ can be treated analogously. Without loss of generality we may further assume that

$$\alpha = 0 \quad . \tag{209}$$

We shall refer to this boundary value problem as the problem (L).

We now introduce the new independent variable

$$z = \rho x \quad , \tag{210}$$

and transform the boundary value problem (L) into an equivalent problem in z, to which we shall refer as the problem $(\tilde{L})$. Let $U(x,\rho)$ be the solution of the problem (L). $U(x,\rho)$ or some of its derivatives will have a boundary layer at $x = \alpha = 0$. The function $U(x,\rho)$ is changed, by the transformation (210) into

$$\tilde{U}(z,\rho) = U(\frac{z}{\rho},\rho) \quad . \tag{211}$$

$\tilde{U}(z,\rho)$ is the solution of the problem $(\tilde{L})$. Since

$$\tilde{U}^{(i)}(z,\rho) = \rho^{-i}\, U^{(i)}(x,\rho) \quad , \tag{212}$$

the problem (L) can be written

$$\tilde{L}(y) \equiv y^{(n)} + b_0(\frac{z}{\rho})y^{(m)} + \sum_{\nu=1}^{n} a_\nu(\frac{z}{\rho})\rho^{-\nu}y^{(n-\nu)} + \sum_{\mu=1}^{m} b_\mu(\frac{z}{\rho})\rho^{-\mu}\, y^{(m-\mu)} = 0 \quad , \tag{2.13}$$

$$\tilde{L}_i \equiv \begin{cases} y^{(\lambda_i)}(\rho\beta) = \ell_i \rho^{-\lambda_i} \quad , & 0 < i < r \\ y^{(\tau_i)}(0) = \ell_i \rho^{-\tau_i} \quad , & r+1 < i < n \ . \end{cases} \tag{214}$$

If we let ρ tend to infinity in the coefficients of (213) we obtain the simple "limiting" differential equation

$$y^{(n)} + b_0(0)\, y^{(m)} = 0 \quad . \tag{215}$$

It may be expected that the function $\tilde{U}(x,\rho)$ tends with increasing ρ to a function

$$\tilde{u}(z) = \lim_{\rho\to\infty} \tilde{U}(z,\rho) \tag{216}$$

which satisfies the differential equation (215). Since (215) is of the same order as (213) we *expect that the function* $\tilde{U}(x,\rho)$ will not have a boundary layer for large ρ.

The transformation (210) may be described as a stretching of the function $U(x,\rho)$. If $\tilde{U}(z,\rho)$ does not have a boundary layer, we have, in a way, "stretched out" the boundary layer.

The problem arises then what boundary conditions are satisfied by the limiting function $\tilde{u}(z)$. If the boundary layer of the function $U(x,\rho)$ at $x = 0$ occurs in $U(x,\rho)$ itself and not in a derivative of $U(x,\rho)$, the interpretation of (210) as a stretching which becomes infinite when $\rho \to \infty$, suggests that we have

$$\tilde{u}(\infty) = u(0) \quad ,$$

where $u(0)$ is the value assumed by the function

$$u(x) = \lim_{\rho\to\infty} U(x,\rho) \tag{217}$$

at $x = 0$. $u(0)$ will, in general, be different from the boundary value prescribed for $U(x,\rho)$ at $x = 0$.

The transformation (210) is frequently used for the solution of more complicated boundary layer problems. The following points are then usually taken for granted without proof:

(a) That $\lim_{\rho\to\infty} \tilde{U}(z,\rho)$ exists.

(b) That the limit $\tilde{u}(z)$ satisfies the limiting differential equation

(c) That $\tilde{u}(\infty) = u(0)$.

In our investigation we have been able to find an asymptotic approximation for $U(x,\rho)$ directly, so that we did not need the transformation (210). But we are now able to prove the statements (a),(b), (c), for our problem (L). This is what we are going to do in this §.

38. In chapter I we have derived for $U(x,\rho)$ the following asymptotic representation (compare (171)):

$$U(x,\rho) = [\pi\eta(x)]\,\rho^{-\tau_{r+1}}\, e^{\rho\int_0^x \varphi(\xi)d\xi} + [u(x)] \quad . \tag{218}$$

In this formula we are using the following abbreviations:

$$\pi = \{-b_0(0)\}^{-\tau_{r+1}} \frac{\delta_1}{\delta} \tag{219}$$

where

$$\delta = (-1)^{r+1} \begin{vmatrix} L_1(u_1) & \cdots\cdots & L_1(u_m) \\ & \cdots\cdots\cdots & \\ L_r(u_1) & \cdots\cdots & L_r(u_m) \\ L_{r+2}(u_1) & \cdots\cdots & L_{r+2}(u_m) \\ & \cdots\cdots\cdots & \\ L_n(u_1) & \cdots\cdots & L_n(u_m) \end{vmatrix} \quad . \tag{220}$$

(Compare formula (159)),

$$\delta_1 = \begin{vmatrix} \ell_1 & L_1(u_1) & \cdots & L_1(u_m) \\ \ell_2 & L_2(u_2) & \cdots & L_2(u_m) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \ell_n & L_n(u_1) & \cdots & L_n(u_m) \end{vmatrix} \tag{221}$$

(compare section 18);

$$\varphi(x) = -b_0(x) \tag{222}$$

and

$$u(x) = \sum_{\mu=1}^{m} \frac{\delta_{1+\mu}}{\delta} u_\mu(x) \quad . \tag{223}$$

(Compare section 17 for the $\delta_{1+\mu}$, – we have dropped the index b, used there, as unnecessary here –; and compare section 6, theorem 1 for the $u_\mu(x)$).

In consequence of theorem 1, formulas (119) and (120), we may differentiate (218) formally at least $n-1$ times, i.e.

$$U^{(i)}(x,\rho) = [\pi^{i}(x)\eta(x)]\rho^{i-\tau_{r+1}} e^{\rho \int_0^x \varphi(\xi)d\xi} + [u^{(i)}(x)], \; (i = 0,1,\ldots,n-1) \quad . \tag{224}$$

39. Knowing $U(x,\rho)$ and its derivatives we can now easily calculate the function $\tilde{U}(z,\rho)$ and its derivatives with respect to z. For, considering that

$$e^{\sigma \int_0^{z/\rho} \varphi(\xi)d\xi} = e^{\int_0^z \varphi(\frac{\zeta}{\rho})d\zeta} \tag{225}$$

we find, upon substitution of (212) into (224), that

$$\tilde{U}^{(i)}(z,\rho) = [\pi\varphi^{i}(\tfrac{z}{\rho})\cdot\eta(\tfrac{z}{\rho})]\rho^{-\tau_{r+1}} e^{\int_0^z \varphi(\frac{\zeta}{\rho})d\zeta} + \rho^{-i}(\tfrac{z}{\rho})], \; (i = 0,1,\ldots,n-1) \quad . \tag{226}$$

Thus we have solved the problem $(\tilde{L})$.

The result of the passage to the limit in (226) can be most easily expressed by one formula comprising all cases that can arise, if we introduce the following symbol:

$$\varepsilon(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0 \ . \end{cases} \tag{227}$$

Then we can write

$$\tilde{u}(z) = \lim_{\rho \to \infty} \tilde{U}(z,\rho) = [\pi \cdot \varphi(0)]e^{z\varphi(0)} \varepsilon(\tau_{r+1}) + u(0) \quad . \tag{228}$$

Formula (228) is valid for $0 < z < \infty$.

We can now confirm the three unproved statements (a), (b), (c) of section 37. $\tilde{u}(z)$ does exist, and it satisfies the limiting differential equation (215), as we may readily verify by substitution. In fact, $\tilde{u}(z)$ reduces to a constant, unless $\tau_{r+1} = 0$. Finally we see from (228) that $\tilde{u}(\infty) = u(0)$. This proves the statements (a), (b), (c).

For the limit of the i-th derivative of $\tilde{U}(z,\rho)$ we find, from (226),

$$\lim_{\rho \to \infty} \tilde{U}^{(i)}(z,\rho) = [\pi\varphi^{i}(0)]e^{z\varphi(0)} \varepsilon(\tau_{r+1}) \ , \ (i = 1,2,\ldots,n-1) \quad , \tag{229}$$

which is also the i-th derivative of $\tilde{u}(z)$. We conclude from (228) and (229) that $\tilde{U}(0,\rho)$ and $\tilde{U}^{(i)}(0,\rho)$ remain finite as $\rho \to \infty$, in other words, $\tilde{U}(z,\rho)$ does not have a boundary layer at $x = 0$, the boundary layer has been "stretched out".

40. We have seen that the limiting differential equation (215) is satisfied by the function $\tilde{u}(z)$ obtained by passing to the limit in the solution $\tilde{U}(z,\rho)$ of the problem $(\tilde{L})$. How can we find a complete set of n boundary conditions satisfied by $\tilde{u}(z)$?

If we formally let ρ tend to infinity in the boundary conditions (214), we obtain boundary conditions

$$\tilde{y}^{(\lambda_i)}(\infty) = \ell_i \ \varepsilon(\lambda_i) \quad , \quad (i = 1,2,\ldots,r) \tag{230}$$

$$\tilde{y}^{(\tau_i)}(0) = \ell_i\, \varepsilon(\tau_i) \quad , \quad (i = r+1, r+2, \ldots, n) \quad . \tag{231}$$

From the remark made at the end of the last section it follows immediately that $\tilde{u}(z)$ satisfies the boundary conditions (231). If $\lambda_\gamma = \sigma$, the last boundary conditions (230) can only be satisfied if, by coincidence, $\ell_i = u(0)$. We therefore replace the last boundary condition of (230) by the condition

$$\tilde{u}(\infty) = u(0) \quad ,$$

which we have proved to be satisfied. The other boundary conditions (230) are certainly satisfied, since all the derivatives of $\tilde{u}(z)$ vanish at $z = \infty$.

§3. The Non-Homogeneous Differential Equation $\frac{1}{\rho} N(y) + M(y) = f(x)$.

41. Introduction: It is an open question whether the Main Theorem remains valid in full generality for the *non-homogeneous* differential equation

$$\frac{1}{\rho} N(y) + M(y) = f(x) \quad . \tag{232}$$

But we will be able to answer this question in the affirmative when

$$n - m = 1 \quad , \tag{233}$$

provided none of the boundary conditions not canceled in application of the rule of the Main Theorem involves an order of differentiation greater than $m-1$. The meaning of this latter condition is easily understandable: If one of the uncanceled boundary conditions is of the order $n-1$, then the boundary value problem formed by the limiting differential equation

$$M(y) = f(x)$$

and the uncanceled boundary conditions is of a type to which the usual method of solution by means of the Green's Function cannot be applied, since this method presupposes that the boundary conditions are of lower order of differentiation than the differential equation. In our treatment of the homogeneous differential equation the relative order of differentiation of the limiting differential equation and of the remaining boundary

conditions did not play any important role, and it is by no means certain that the condition above is really necessary in the non-homogeneous case. But it simplifies our proof greatly.

42. The problem defined by (232) and (233) and the boundary conditions (108) will be called the problem (N). If, instead of (108), we prescribe the corresponding homogeneous boundary conditions, we shall speak of the problem (N'). We assume that $0 < r < n$, i.e. we consider only actual boundary conditions leaving aside the initial value problem. (The initial value problem can be treated by the *same method.*) It is easy to extend the proof below to the more general boundary conditions (201).

Let $Z(x,\rho)$ be the solution of (N), $z(x,\rho)$ the solution of (N') and $U(x,\rho)$ the solution of the homogeneous differential equation (101) satisfying the non-homogeneous boundary conditions (108). Then

$$Z(x,\rho) = z(x,\rho) + U(x,\rho) \quad . \tag{234}$$

Since the asymptotic behavior of $U(x,\rho)$ for large has already been investigated in chapter I it is sufficient to discuss *the problem* (N').

43. The Green's Function:

It is known that the function $z(x,\rho)$ can be written in the form

$$z(x,\rho) = \int_{\alpha}^{\beta} G(x,t,\rho)\ \rho\ f(t)\ dt \quad . \tag{235}$$

The "*Green's Function*" $G(x,t,\rho)$ can be constructed in the following manner:

Let $y_1(x,\rho)$, $y_2(x,\rho) \cdots y_n(x,\rho)$ be a fundamental system of solutions of $\frac{1}{\rho} N(y) + M(y) = 0$, and set

$$h(t,\rho) = \begin{vmatrix} y_1^{(n-1)}(t,\rho) & y_2^{(n-1)}(t,\rho) & \cdots & y_n^{(n-1)}(t,\rho) \\ y_1^{(n-2)}(t,\rho) & y_2^{(n-2)}(t,\rho) & \cdots & y_n^{(n-2)}(t,\rho) \\ \cdots & \cdots & \cdots & \cdots \\ y_1(t,\rho) & y_2(t,\rho) & \cdots & y_n(t,\rho) \end{vmatrix} .$$

$$k(x,t,\rho) = \operatorname{sgn}(x-t) \begin{vmatrix} y_1(x,\rho) & y_2(x,\rho) & \cdots & y_n(x,\rho) \\ y_1^{(n-2)}(t,\rho) & y_2^{(n-2)}(t,\rho) & \cdots & y_n^{(n-2)}(t,\rho) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1(t,\rho) & y_2(t,\rho) & \cdots & y_n(t,\rho) \end{vmatrix} \tag{236}$$

and

$$g(x,t,\rho) = \frac{1}{2}\,\frac{k(x,t,\rho)}{h(t,\rho)} . \tag{237}$$

Then

$$G(x,t,\rho) = (-1)^n\,\frac{H(x,t,\rho)}{\Delta(\rho)} \tag{238}$$

where

$$H(x,t,\rho) = \begin{vmatrix} y_1(x,\rho) & y_2(x,\rho) & \cdots & y_n(x,\rho) & g(x,t,\rho) \\ L_1(y_1) & L_1(y_2) & \cdots & L_1(y_n) & L_1(g) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(y_1) & L_n(y_2) & \cdots & L_n(y_n) & L_n(g) \end{vmatrix} \tag{239}$$

and

$$\Delta(\rho) = \begin{vmatrix} L_1(y_1) & L_1(y_2) & \cdots & L_1(y_n) \\ L_2(y_1) & L_2(y_2) & \cdots & L_2(y_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ L_n(y_1) & L_n(y_2) & \cdots & L_n(y_n) \end{vmatrix} \quad . \tag{240}$$

$L_i(g)$, $i = 1,2,\ldots,n$, means that the operator L_i is applied to $g(x,t,\rho)$ considered a function of x.

44. The asymptotic value of the Green's Function:

We take as the fundamental system $y_i(x,\rho)$ the functions (114), (115) specialized for $n-m = 1$, i.e.

$$\left.\begin{aligned} y_1(x,\rho) &= V(x,\rho) = e^{\rho \int_a^x \varphi(\xi)d\xi} [\eta(x)] \\ y_{1+\mu}(x,\rho) &= U_\mu(x,\rho) = [u_\mu(x)] \qquad (\mu = 1,2,\ldots,n-1), \end{aligned}\right\} \tag{241}$$

where

$$\varphi(x) = -\, b_0(x) \quad .$$

(Note that our notations differ somewhat from those of chapter I and of chapter II, §1.)

We have then

$$L_i(g) = \frac{1}{2} \frac{L_i(k)}{h(t,\rho)}$$

where

$$L_i(k) = \begin{vmatrix} L_i'(V), & L_i'(U_1), & \cdots & L_i'(U_{n-1}) \\ V^{(n-2)}(t,\rho) & U_1^{(n-2)}(t,\rho) & \cdots & U_{n-1}^{(n-2)}(t,\rho) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ V(t,\rho) & U_1(t,\rho) & \cdots & U_n(t,\rho) \end{vmatrix} \tag{242}$$

with

$$L_i' = L_i \quad , \quad \text{for} \quad i < r$$

$$L_i' = -L_i \quad , \quad \text{for} \quad i > r \quad .$$

Let us assume that

$$b_0(x) < 0 \quad .$$

Then

$$L_i'(V) = \begin{cases} \rho^{\lambda_i}[\varphi^{\lambda_i}(\beta)\ \eta(\beta)]e^{\rho w} \ , & \text{for} \quad i < r \\ -\rho^{\tau_i}[\varphi^{\tau_i}(\alpha)\ \eta(\alpha)] \ , & \text{for} \quad i > r \end{cases} \tag{243}$$

where

$$w = \int_\alpha^\beta \varphi(\xi) d\xi \quad .$$

If we expand (242) in terms of its first column, we see that in $\alpha < t < \beta$ the 3rd, 4th, etc. terms of the expansion are of lower order than the second term. Hence

$$L_i(k) = L_i'(V)[\bar{h}(t)] - V^{(n-2)}(t,\rho)\ [L_i(\bar{k})] \quad , \tag{244}$$

where

$$\bar{h}(t) = \begin{vmatrix} u_1^{(n-2)}(t) & u_2^{(n-2)}(t) & \cdots & u_{n-1}^{(n-2)}(t) \\ u_1^{(n-3)}(t) & u_2^{(n-3)}(t) & \cdots & u_{n-1}^{(n-3)}(t) \\ \cdots & \cdots & \cdots & \cdots \\ u_1(t) & u_2(t) & \cdots & u_{n-1}(t) \end{vmatrix} \tag{245}$$

and

$$\bar{k}(x,t) = \text{sgn}(x-t) \begin{vmatrix} u_1(x) & u_2 & \cdots & u_{n-1}(x) \\ u_1^{(n-3)}(t) & u_2^{(n-3)}(t) & \cdots & u_{n-1}^{(n-3)}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ u_1(t) & u_2(t) & \cdots & u_{n-1}(t) \end{vmatrix} . \tag{246}$$

Expanding $h(t,\rho)$ in terms of its first column we obtain similarly

$$h(t,\rho) = V^{(n-1)}(t,\rho)\ [\bar{h}(t)], \quad \text{for} \quad \alpha < t < \beta \ . \tag{247}$$

$\bar{h}(t) \neq 0$, because otherwise $u_2(t), u_3(t), \dots, u_{n-1}(t)$ would be linearly dependent.

From (237), (243), (244) and (247) we find

$$L_1(g(x,t,\rho)) = \frac{L_1'(V)}{2V^{(n-1)}(t,\rho)}\ [1] - \frac{1}{\rho\varphi(t)} \tag{248}$$

where

$$\bar{g}(x,t) = \frac{\bar{k}(x,t)}{2\bar{h}(t)} \ . \tag{249}$$

Furthermore, expansion of (236) in terms of its first column yields

$$k(x,t,\rho) = \operatorname{sgn}(x-t)\ V(x,\rho)\ [\bar{h}(t)] - V^{(n-2)}(t,\rho)\ [\bar{k}(t)]$$

and therefore, because of (237) and (247)

$$g(x,t,\rho) = \operatorname{sgn}(x-t)\ \frac{V(x,\rho)}{2V^{(n-1)}(t,\rho)} - \frac{1}{\rho\varphi(t)}\ [\bar{g}(x,t)] \ . \tag{250}$$

Now we substitute (250) and (248) into (239) and see that

$$H(x,t,\rho) = H_1(x,t,\rho) + H_2(x,t,\rho) \tag{251}$$

where

$$H_1(x,t,\rho) = \frac{1}{2V^{(n-1)}(t,\rho)} \begin{vmatrix} V(x,\rho) & [u_1(x)] & \cdots & [u_{n-1}(x)], & \operatorname{sgn}(x-t)\ V(x,\rho) \\ L_1(V) & [L_1(u_1)] & \cdots & [L_1(u_{n-1})] & L_1'(V) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(V) & [L_n(u_1)] & \cdots & [L_n(u_{n-1})] & L_n'(V) \end{vmatrix} \tag{252}$$

and

$$H_2(x,t,\rho) = -\frac{1}{\rho\ (t)} \begin{vmatrix} V(x,\rho) & [u_1(x)] & \cdots & [u_{n-1}(x)] & [\bar{g}(x,t)] \\ L_1(V) & [L_1(u_1)] & \cdots & [L_1(u_{n-1})] & [L_1(\bar{g})] \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ L_n(V) & [L_n(u_1)] & & [L_n(u_{n-1})] & (L_n(\bar{g})] \end{vmatrix} \tag{253}$$

The asymptotic expression for $\Delta(\rho)$ is (compare chapter I)

$$\Delta(\rho) = L_1(V)[\bar{\Delta}] = [\varphi_1^{\lambda_1}(\beta)\ \eta(\beta)\ \bar{\Delta}]e^{\rho w} \tag{254}$$

where

$$\bar{\Delta} = \begin{vmatrix} L_2(u_1) & \cdots & L_2(u_{n-1}) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ L_n(u_1) & \cdots & L_n(u_{n-1}) \end{vmatrix} \quad . \tag{255}$$

As in chapter I we make the assumption

$$\bar{\Delta} \neq 0 \quad . \tag{256}$$

($\bar{\Delta}$ is identical with δ_b of formula (159) for this particular case.)

45. Corresponding to the representation of $H(x,t,\rho)$ as a sum of two terms in (251) we find, upon substitution of (251) into (238),

$$G(x,t\rho) = G_1(x,t,\rho) + G_2(x,t,\rho) \tag{257}$$

with

$$G_1(x,t,\rho) = (-1)^n \frac{H_1(x,t\rho)}{\Delta(\rho)} \tag{258}$$

$$G_2(x,t,\rho) = (-1)^n \frac{H_2(x,t,\rho)}{\Delta(\rho)} \quad . \tag{259}$$

Using (235) this leads to a representation of the solution $z(x,\rho)$ as a sum of two integrals:

$$z(x,\rho) = z_1(x,\rho) + z_2(x,\rho) \tag{260}$$

with

$$z_1(x,\rho) = \int_\alpha^\beta G_1(x,t\rho)\ \rho\ f(t)dt \tag{259}$$

$$z_2(x,\rho) = \int_\alpha^\beta G_2(x,t,\rho)\ \rho\ f(t)dt\ . \tag{260}$$

We shall prove that, as ρ tends to infinity, $z_1(x,\rho)$ tends to zero, while $z_2(x,\rho)$ tends to a solution of $M(y) = f(x)$.

46. In this section we are going to show that

$$\lim_{\rho\to\infty} z_1(x,\rho) = \lim_{\rho\to\infty} \int_\alpha^\beta G_1(x,t,\rho)\ \rho\ f(t)dt = 0\ . \tag{261}$$

To this end we write (259) in the form

$$z_1(x,\rho) = \int_\alpha^x G_1(x,t,\rho)\ \rho\ f(t)dt + \int_x^\beta G_1(x,t,\rho)\ \rho\ f(t)dt \tag{262}$$

and prove that each of the two integrals in (262) tends to zero.

a) In the first integral of (262) we have $t < x$. Hence, $G_1(x,t,\rho)$ has to be determined with $+V(x,\rho)$ as the last term of the first row of the determinant in (252). In order to find an asymptotic expression for $H_1(x,t,\rho)$ we expand the determinant in (252) in terms of the minors formed by its first and last columns. These minors are either zero or of the form

$$\left.\begin{array}{c} \pm 2\ L_\gamma(V)\ L_\varepsilon(V) \\ \text{or} \\ \pm 2\ V(x,\rho) L_\varepsilon(V) \end{array}\right\} \qquad \begin{array}{l} \gamma < r \\ \varepsilon > r \end{array} \tag{263}$$

i.e. they are of the form

$$\left.\begin{array}{ll} \rho^{\lambda_\gamma + \tau_\varepsilon}\ e^{\rho w} & [q] \\ \quad\text{or} & \\ \rho^{\tau_\varepsilon}\ e^{\sigma \int_\alpha^x \varphi(\xi)d\xi} & [q'] \end{array}\right\} \tag{264}$$

where q and q' are certain numbers different from zero. Since λ_1 and τ_{r+1} are greater than all the other λ_i's and τ_i's, it follows from (264) that, in $\alpha < x < \beta$, the expression of highest order of magnitude among all the minor (263) is $\pm 2\, L_1(V)\, L_{r+1}(V)$, and no other minor has the same order. Hence, (252) can be written, - in $\alpha < x < t$ -, as follows:

$$H_1(x,t,\rho) = (-1)^{n-r} \rho^{\tau_{r+1}-n+1} L_1(V) e^{-\rho \int_\alpha^t \varphi(\xi)d\xi} \left[\frac{\varphi^{\tau_{r+1}}(\alpha)\ \eta(\alpha)}{\varphi^{n-1}(t)\ \eta(t)}\, v(x)\right] \qquad (265)$$

where

$$v(x) = \begin{vmatrix} [u_1(x)] & \cdots\cdots\cdots & [u_{n-1}(x)] \\ [L_2(u_1) & \cdots\cdots\cdots & [L_2(u_{n-1}] \\ \cdots & \cdots\cdots\cdots & \cdots \\ [L_r(u_1)] & \cdots\cdots\cdots & [L_r(u_{n-1})] \\ [L_{r+2}(u_1) & \cdots\cdots\cdots & [L_{r+2}(u_{n-1})] \\ \cdots & \cdots\cdots\cdots & \cdots \\ [L_n(u_1)] & \cdots\cdots\cdots & [L^n(u_{n-1})] \end{vmatrix} \qquad (266)$$

is the cofactor of the minor $\pm 2\, L_1(V)\, L_{r+1}(v)$ in (252). (265), (258) and (254) show that for $t < x$,

$$G(x,t,\rho) = (-1)^r \rho^{\tau_{r+1}-n+1} e^{-\rho \int_\alpha^t \varphi(\xi)d\xi} \left[\frac{\varphi^{\tau_{r+1}}(\alpha)\ \eta(a)\ v(x)}{\varphi^{n-1}(t)\ \eta(t)\ \bar{\Delta}}\right] \qquad (267)$$

and therefore

$$\int_\alpha^x G(x,t,\rho)\rho\, f(t)dt = (-1)^{\tau_{r+1}-n+2} \varphi^{\tau_{r+1}}(\alpha)\ \eta(\alpha)\ \frac{v(x)}{\bar{\Delta}} \int_\alpha^t \varphi(\xi)d\xi\ F(t,\rho) \qquad (268)$$

with

$$F(t,\rho) = \frac{f(t)}{[\varphi^{n-1}(t)\ \eta(t)]} .$$

The integral in the right hand member of (268) tends to zero as $\rho \to \infty$, since its integrand tends to zero in the interior of the interval and remains bounded at the endpoints. $\delta^{\tau_{r+1}-n+2}$ remains bounded, because we have assumed that

$$\tau_{r+1} < n-2 .$$

Consequently, the left member of (268) tends to zero, as $\rho \to \infty$.

b) In the second integral of (262) we have $t > x$. Hence, in the determinant in (252) the last term of the first row is $-V(x,\rho)$. Expanding this determinant in terms of the minors formed by its first and last columns we see that this time the minor of highest order is formed by the two first rows of the determinant. The value of this minor is

$$\pm 2\ V(x,\rho)\ L_1(v) .$$

A calculation analogous to that used in part a) of this section leads to

$$\int_x^\beta G_1(x,t,\rho)\rho\ f(t)dt = \rho^{-n+2}\ [\frac{\eta(x)\ Q}{\bar{\Delta}}] \tag{269}$$

where the constant Q is the limit of the cofactor of the minor above. Since $t > x$ in the integral in the right member of (269), the integrand of that integral tends to zero in the interior of the interval of integration, $x < t < \beta$, as ρ approaches infinity. Furthermore, the integrand is bounded at the endpoints of the interval of integration. Consequently, the integral tends to zero. On the other hand, our assumption $m > 0$ implies $n-2 > 0$, and the power of ρ in (269) is therefore not positive. Hence, the left member of (269) tends to zero, as $\rho \to \infty$.

This completes the proof of (261).

47. We now turn to the asymptotic calculation of $z_2(x,\rho)$, (see (260)). We note first, on expanding the determinant in (253) with respect to its first column, that $H_2(x,t,\rho)$ can be written, in $\alpha < x < \beta$, in the form

$$H_2(x,t,\rho) = \frac{L_1(V)}{\rho\ \varphi(t)} [\bar{H}(x,t)] \tag{270}$$

where

$$\bar{H}(x,t) = \begin{vmatrix} u_1(x) & \cdots\cdots & u_{n-1}(x) & \bar{g}(x,t) \\ L_2(u_1) & \cdots\cdots & L_2(u_{n-1}) & L_2(\bar{g}) \\ \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots\cdots & \cdots & \cdots \\ L_n(u_1) & \cdots\cdots & L_n(n_{n-1}) & L_n(\bar{g}) \end{vmatrix} \tag{271}$$

is the limit of the cofactor of the element $L_1(V)$ in (253). Substituting (270) in (259) it follows that

$$G_2(x,t,\rho) = (-1)^n \frac{1}{\rho} \frac{\bar{H}(x,t)}{\varphi(t)\cdot\bar{\Delta}}$$

and this formula, together with (260) gives

$$z_2(x,\rho) = (-1)^n \int_\alpha^\beta [\frac{\bar{H}(x,t)}{\varphi(t)\bar{\Delta}}] f(t)dt \quad .$$

In this expression we may pass to the limit under the integral sign, since the asymptotic expression for the integrand is valid in the whole interval $\alpha < t < \beta$. Thus we obtain, replacing at the same time $\varphi(t)$ by its value $-b_0(t)$,

$$\lim_{\rho\to\infty} z_2(x,\rho) = \int_\alpha^\beta (-1)^{n-1} \frac{\bar{H}(x,t)}{\bar{\Delta}} \frac{f(t)}{b_0(t)} dt \quad . \tag{272}$$

Now we combime (272) with (261) and (260), and conclude that the right member of (272) is the value of $\lim_{\rho \to \infty} z(x,t,\rho)$. But the definition of $\overline{H}(x,t)$ and $\overline{\Delta}$ in (271) and (255), respectively, show us that

$$(-1)^{n-1} \frac{\overline{H}(x,t)}{\overline{\Delta}}$$

is exactly the Green's Function belonging to the differential expression

$$\frac{1}{b_0(x)} M(y)$$

and to the boundary conditions $L_2 = 0$, $(i = 2,3,\ldots,n)$.

Since these boundary conditions do not involve derivatives of higher than $(n-2)$nd order, this proves that $z(x,\rho)$ tends to a solution of the limiting differential equation, satisfying all boundary conditions, except the first one.

48. We summarize the results of this § in the following theorem.

Theorem 5:

We consider the problem (N) defined by the differential equation

$$\frac{1}{\rho} N(y) + M(y) = f(x) \qquad 232)$$

and the boundary conditions (108). $N(y)$ and $M(y)$ are differential expressions of the form (103) and (104) with

$$n-m=1 \quad .$$

We make the following assumptions:

(a) Conditions 1° - 5°, 7° and 8° of the Main Theorem (section 4) are satisfied.

(b) $b_0(x) < 0$ in $\alpha \leq x \leq \beta$.

(c) $\tau_{r+1} < n-1$.

(d) $0 < r < n$.

(e) $f(x)$ is integrable in $\alpha \leq x \leq \beta$.

Then the solution $Z(x,\rho)$ of the problem (N) tends - as ρ approaches infinity - to a solution of the differential equation

$$M(y) = f(x)$$

satisfying all the boundary conditions (108) except, in general, the first one, $L_1(y)$.

A strictly analogous theorem holds for $b_0(x) > 0$.

Chapter III

SOME RELATED PROBLEMS

§1. An Example for Boundary Layer Problems in Systems of Differential Equatins.

49. A great number of unsolved boundary layer problems with important applications can be formulated for systems of ordinary differential equations. We are going to discuss in this section a very elementary example in order to give an idea of the boundary layer phenomena that can arise for systems:

We shall discuss the system

$$\begin{aligned} \rho^{-1}u'' &= a\,u + b\,v \\ v'' &= c\,u + d\,v \end{aligned} \tag{301}$$

with constant $a, b, c, d,$ assuming that

$$a \neq 0 \quad . \tag{302}$$

As boundary conditions we prescribe

$$u(\alpha) = u_\alpha,\ v(\alpha) = v_\alpha,\ u(\beta) = u_\beta,\ v(\beta) = v_\beta \tag{303}$$

where $u_\alpha, v_\alpha, u_\beta, v_\beta$ are constants.

The "limiting problem", obtained by setting $\rho^{-1} = 0$ in (301) is equivalent to the differential equation of second order

$$v'' - \frac{\delta}{a} v = 0 \tag{304}$$

where

$$\delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad , \tag{305}$$

and to the relation

$$u = -\frac{b}{a} v \quad . \tag{306}$$

The following questions arise:

(a) Do the solutions $U(x,\rho)$, $V(x,\rho)$ of (301) and (303) converge as $\rho \to \infty$?

(b) What boundary conditions do the limit functions $\bar{U}(x)$ and $\bar{V}(x)$ satisfy if they exist, and are they solutions of the limiting differential equation (304)?

The functions $\bar{U}(x)$ and $\bar{V}(x)$ cannot be expected to satisfy all four boundary conditions (303) and also the condition (306), for the prescribed boundary values (303) will, in general, not satisfy the condition (306). The answer to these questions is supplied by the following Theorem:

<u>Theorem 6:</u>

Let $u = U(x,\rho)$, $v = V(x,\rho)$ be solutions of the system of differential equations

$$\left.\begin{aligned} \rho^{-1}u'' &= a\,u + b\,v \\ v'' &= c\,u + d\,v \end{aligned}\right\} \tag{301}$$

(a, b, c, d constants) satisfying the boundary conditions

$$u(\alpha) = u_\alpha,\ v(\alpha) = v_\alpha,\ u(\beta) = u_\beta,\ v(\beta) = v_\beta \tag{303}$$

where u_α, u_β, v_α, v_β are constants. Let us further assume:

Assumption 1: $a \neq 0$.

Assumption 2: a, b, c, d, α and β are given in such a way that the differential equation

$$y'' - \frac{\delta}{a}\,y = 0 \tag{307}$$

with

$$\delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{305}$$

and the boundary conditions

$$y(\alpha) = 0 \quad , \quad y(\beta) = 0$$

can be satisfied by the function $y(x) \equiv 0$ only.

Then we can make the following statements:

(A) The function $V(x,\rho)$ converges, as $\rho \to \infty$, to a function $\bar{V}(x)$, satisfying the differential equation (307) and having the boundary values

$$\bar{V}(\alpha) = v_\alpha \quad , \quad \bar{V}(\beta) = v_\beta \tag{308}$$

(B) If $a > 0$, then the function $U(x,\rho)$ converges in $\alpha < x < \beta$, as $\rho \to \infty$, to the function

$$\bar{U}(x) = -\frac{b}{a}\bar{V}(x) \tag{309}$$

(which, of course, satisfies the differential equation (307)).

(C) If $a < 0$, then the function $U(x,\rho)$ does not converge but remains bounded, as $\rho \to \infty$, except when the prescribed boundary values satisfy the condition

$$\begin{aligned} a\,u_\alpha + b\,v_\alpha &= 0 \\ a\,u_\beta + b\,v_\beta &= 0 \quad , \end{aligned} \tag{310}$$

in which case statement (B) remains true for $a < 0$ also.

50. To prove this theorem we start from the observation that $U(x,\rho)$ and $V(x,\rho)$ are both solutions of the differential equation

$$\rho^{-1}\{y^{(4)} - d\,y''\} - a\,y'' + \delta\,y = 0 \quad . \tag{311}$$

In fact, if we multiply the second differential equation of (301) by b and substitute into it the expressions

$$b\,v = \rho^{-1}u'' - a\,u \quad , \quad b\,v'' = \delta^{-1}u^{(4)} - a\,u''$$

obtained from the first differential equation, we find the following differential equation in u alone,

$$u^{(4)} - (\rho\, a + d)\, u'' + \rho\, \delta\, u = 0 \quad ,$$

which is equivalent with (311). Similarly, it can be shown that (311) is satisfied by $V(x,\rho)$, by eliminating u from the differential equations (301). It is also easily seen that the boundary conditions satisfied by $U(x,\rho)$ and $V(x,\rho)$, considered as solutions of (311), are, respectively,

for $U(x,\rho)$		for $V(x,\rho)$	
$y(\alpha) = u_\alpha$		$y(\alpha) = v_\alpha$	
$y(\beta) = u_\beta$	(312a)	$y(\beta) = v_\beta$	(312b)
$y''(\alpha) = (a\, u_\alpha + b\, v_\alpha)\rho$		$y''(\alpha) = c\, u_\alpha + d\, v_\alpha$	
$y''(\beta) = (a\, u_\beta + b\, v_\beta)\rho$		$y''(\beta) = c\, u_\beta + d\, v_\beta$.	

The boundary conditions for $V(x,\rho)$ are of the type considered in our Main Theorem (chapter I, section 3). Applying the Main Theorem for these boundry conditions and for $n-m = 2$, $b_0(x) = -a$, we see that for $a > 0$ as well as for $a < 0$ the function $V(x,\rho)$ converges to that solution of the limiting differential equation (307) which satisfies the boundary conditions $y(\alpha) = v_\alpha$, $y(\beta) = v_\beta$. This completes the proof for statement (A).

51. For $U(x,\rho)$ a special calculation is necessary, since the boundary conditions for $U(x,\rho)$ contain ρ, a case not considered in our Main Theorem. Our principal tool in chapter I, the asymptotic representation of a fundamental system of solutions of the given differential equation, can be applied to the differential equation (311) and yields then (see theorem 1, section 6)

$$\begin{aligned} U_1(x,\rho) &= e^{\sigma\sqrt{a}(x-\alpha)} \quad [1] \\ U_2(x,\rho) &= e^{-\sigma\sqrt{a}(x-\alpha)} \quad [1] \qquad\qquad (313) \\ U_3(x,\rho) &= [A(x)] \\ U_4(x,\rho) &= [B(x)] \quad . \end{aligned}$$

Here the functions $A(x)$, $B(x)$ form a fundamental system of solutions of the limiting differential equation (307) and

$$\sigma = \sqrt{\rho} \ .$$

With these functions we repeat the general reasoning of chapter I for this special case. We first re-write the boundary conditions (312a) in a form somewhat more convenient for our purpose:

$$\begin{aligned} \rho^{-1}y''(\beta) &= \ell_1 = a\,u_\beta + b\,v_\beta \ , & \rho^{-1}y''(\alpha) &= \ell_3 = a\,u_\alpha + b\,v_\alpha \\ y(\beta) &= \ell_2 = u_\beta \ , & y(\alpha) &= \ell_4 = u_\alpha \ . \end{aligned} \tag{314}$$

Let us further introduce the abbreviations

$$e^{\sqrt{a}(\beta-\alpha)} = W_1 \ , \ e^{-\sqrt{a}(\beta-\alpha)} = W_2 \ . \tag{315}$$

<u>First case:</u> $a > 0$.

Defining $\Delta(\rho)$ as in (134) we find

$$\Delta(\rho) = \begin{vmatrix} [a]W_1 & [a]W_2 & \rho^{-1}[A''(\beta)] & \rho^{-1}[B''(\beta)] \\ [1]W_1 & [1]W_2 & [A(\beta)] & [B(\beta)] \\ [a] & [a] & \rho^{-1}[A''(\alpha)] & \rho^{-1}[B''(\alpha)] \\ [1] & [1] & [A(\alpha)] & [B(\alpha)] \end{vmatrix} \ . \tag{316}$$

We expand this determinant in terms of its last two rows:

$$\Delta(\rho) = -[a^2] \begin{vmatrix} A(\beta) & B(\beta) \\ A(\alpha) & B(\alpha) \end{vmatrix} W_1 = -[a^2]D\,W_1 \tag{317}$$

with

$$D = \begin{vmatrix} A(\beta) & B(\beta) \\ A(\alpha) & B(\alpha) \end{vmatrix} \ . \tag{318}$$

In consequence of assumption 2 of the theorem to be proved we have

$$D \neq 0 \quad . \tag{319}$$

Similarly, we find

$$\Delta_1(\rho) = \begin{vmatrix} \ell_1 & [a]W_2 & \rho^{-1}[A''(\beta)] & \rho^{-1}[B''(\beta)] \\ \ell_2 & [1]W_2 & [A(\beta)] & [B(\beta)] \\ \ell_3 & [a] & \rho^{-1}[A''(\alpha)] & \rho^{-1}[B''(\alpha)] \\ \ell_4 & [1] & [A(\alpha)] & [B(\alpha)] \end{vmatrix} = -[a\ \ell_1\ D] \tag{320}$$

and

$$\Delta_2(\rho) = \begin{vmatrix} [a]W_1 & \ell_1 & \rho^{-1}[A:(\beta)] & \rho^{-1}[B''(\beta)] \\ [1]W_1 & \ell_2 & [A(\beta)] & [B(\beta)] \\ [a] & \ell_3 & \rho^{-1}[A''(\alpha)] & \rho^{-1}[B''(\alpha)] \\ [1] & \ell_4 & [A(\alpha)] & [B(\alpha)] \end{vmatrix} = -[a\ \ell_3\ D] \quad . \tag{321}$$

In the determinant

$$\Delta_3(\rho) = \begin{vmatrix} [a]W_1 & [a]W_2 & \ell_1 & \rho^{-1}[B''(\beta)] \\ [1]W_1 & [1]W_2 & \ell_2 & [B(\beta)] \\ [a] & [a] & \ell_3 & \rho^{-1}[B''(\alpha)] \\ [1] & [1] & \ell_4 & [B(\alpha)] \end{vmatrix}$$

we subtract the a^{-1} fold of the first row from the second row and the a^{-1} fold of the third row from the fourth, thus obtaining

$$\Delta_3(\rho) = \begin{vmatrix} [a]W_1 & [a]W_2 & \ell_1 & \rho^{-1}[B''(\beta)] \\ [0]W_1 & [0]W_2 & -\frac{b}{a}v_\alpha & [B(\beta)] \\ [a] & [a] & \ell_3 & \rho^{-1}[B''(\alpha)] \\ [0] & [0] & -\frac{b}{a}v_\alpha & [B(\alpha)] \end{vmatrix} \quad . \tag{321}$$

Here we have used the fact that, in consequence of (313)

$$\ell_2 - \frac{\ell_1}{a} = -\frac{b}{\alpha} v_\beta \quad , \quad \ell_4 - \frac{\ell_3}{a} = -\frac{b}{a} v_\alpha \quad . \tag{322}$$

Expanding (321) we find

$$\Delta_3(\rho) = \begin{vmatrix} -\frac{b}{a} v_\beta & B(\beta) \\ -\frac{b}{a} v_\alpha & B(\alpha) \end{vmatrix} \quad [a^2]W_1 \quad . \tag{323}$$

Similarly, we prove that

$$\begin{vmatrix} A(\beta) & -\frac{b}{a} v_\beta \\ A(\alpha) & -\frac{b}{a} v_\alpha \end{vmatrix} \quad [a^2]W_1 \quad . \tag{324}$$

52. From (313), (317), (320) and (321) it follows that

$$\left.\begin{aligned} \lim_{\rho\to\infty} \frac{\Delta_1(\rho)}{\Delta(\rho)} U_1(x,\rho) = 0 \\ \lim_{\rho\to\infty} \frac{\Delta_2(\rho)}{\Delta(\rho)} U_2(x,\rho) = 0 \end{aligned}\right\} \text{ in } \quad \alpha < x < \beta \quad .$$

(323) and (324) show then that

$$\bar{U}(x) = \lim_{\rho\to\infty} U(x,\rho)$$

is that solution of (307) which assumes the boundary values

$$\bar{U}(\alpha) = -\frac{b}{a} v_\alpha \quad , \quad \bar{U}(\beta) - \frac{b}{a} v_\beta \quad .$$

This proves statement (B) of theorem 6.

<u>Second case:</u> $a < 0$.

If $a < 0$, then the absolute values of W_1 and W_2 oscillate with increasing ρ without tending to a limit. In this case we obtain from (316) the asympotic expression

$$[D \quad A^2]\ (W_1 - W_2) \tag{325}$$

instead of (317). For $\Delta_1(\rho)$ and $\Delta_2(\rho)$ we have

$$\Delta_1(\rho) = -[D \quad a]\ (\ell_1 - \ell_3 W_2) \tag{326}$$

$$\Delta_2(\rho) = -[D \quad a]\ (\ell_3 W_1 - \ell_1) \tag{327}$$

and for $\Delta_3(\rho)$ and $\Delta_4(\rho)$

$$\Delta_3(\rho) = -[a^2] \begin{vmatrix} -\frac{b}{a} v_\beta & B(\beta) \\ -\frac{b}{\alpha} v_\alpha & B(\alpha) \end{vmatrix} (W_1 - W_2) \tag{328}$$

$$\Delta_4(\rho) = -[a^2] \begin{vmatrix} A(\beta) & -\frac{b}{a} v_\beta \\ A(\alpha) & -\frac{b}{a} v_\alpha \end{vmatrix} (W_1 - W_2)\ . \tag{329}$$

53. From these expressions it follows immediately that

$$U(x,\rho) = \sum_{i=1}^{4} \frac{\Delta_i(\rho)}{\Delta(\rho)} U_i(x,\rho)$$

does not converge in this case. For

$$\frac{1}{\Delta(\rho)} (\Delta_3(\rho)\ U_3(x,\rho) + \Delta_4(\rho)\ U_4(x,\rho)\ )$$

converges to the same solution of (307) as in the case $a < 0$, while the expression

$$\frac{1}{\Delta} (\Delta_1 U_1 + \Delta_2 U_2) = \frac{\ell_1 - \ell_3 W_2}{a(W_1 - W_2)} e^{i\sqrt{|a|}(x-\alpha)} + \frac{\ell_3 W_1 - \ell_1}{a(W_{1-W_2})} e^{-i\sigma\sqrt{|a|}(x-\alpha)}$$

does not converge unless $\ell_1 = \ell_3 = 0$, in which case it is identically zero.

This completes the proof of theorem 6.

If assumption 2 of theorem 6 is not satisfied, then our reasoning does not hold any more. In that case it would be necessary to take into consideration also the second terms of the asymptotic series used, in order to find the order of magnitude of $\Delta(\rho)$.

Assumption 2 is easily seen to be equivalent, in this case, with

$$\delta \neq 0$$

and

$$\sqrt{-\frac{\delta}{a}}\,(\beta-\alpha) \neq N\pi \quad \text{when} \quad \frac{\delta}{a} < 0$$

where N is any positive or negative integer.

The case $a = 0$ could be easily treated by the same method.

The boundary layers in this § occur only for the function $U(x,\rho)$, which with increasing ρ tends to a function which does not have the prescribed boundary values, except, when these boundary values satisfy the condition (310).

A more adequate and general treatment of boundary layer problems in systems of ordinary linear differential equations could probably be based on the asymptotic solution of linear systems as developed by Langer and G. D. Birkhoff [5]. The assumptions of that theory would, however, have to be generalized for this purpose.

§2. An Example for Boundary Layer Problems with Singularities in the Interior.

54. Introduction: If the assumption 6° of the Main Theorem in chapter I is dropped, i.e. if we admit zeros of $b_0(x)$ in $\alpha < x < \beta$, then our whole theory becomes invalid. For the zeros of $b_0(x)$ are usually singularities of the limiting differential equation $M(y) = 0$, and the theory of the asymptotic solution of differential equations, which was our main tool, fails in this case.

The general treatment of boundary layer problems in this case would probably require an entirely new approach. But it is already interesting to investigate a very simple special problem of this type in which the differential equation can be solved explicitly. We shall discuss the boundary layer problem of the differential equation

$$\frac{1}{\rho} y'' + b(x)y' = f(x) \tag{330}$$

with the boundary conditions

$$y(\alpha) = \ell_2 \quad , \quad y(\beta) = \ell_1 \quad . \tag{331}$$

We make the following assumptions:

1. $b(x)$ is regular analytic in $\alpha \leq x \leq \beta$.
2. $f(x)$ is regular analytic in $\alpha \leq x \leq \beta$.
3. $b(x)$ has a zero at the interior point $x = r$ of the interval $\alpha < x < \beta$. But $b'(r) \neq 0$, and $b(x)$ does not have any other roots in $\alpha \leq x \leq \beta$. (This number r has, of course, nothing to do with the number r used in the first two chapters.)

Assumptions 1 and 2 are by no means essential. We introduce them only in order to simplify our reasoning.

55. The solution of the boundary value problem for the differential equation (330): To simplify the calculations it shall be assumed that the boundary values are

$$y(\alpha) = y(\beta) = 0 \quad .$$

The case of non-homogeneous boundary conditions, which does not add any new features to the problem, is discussed in section 59.

The general solution of (330), as obtained by elementary methods can be written in the form

$$U(x,\rho) = \int_\lambda^x d\xi \int_\mu^\xi \rho\, f(\eta) e^{-\rho(\Lambda(\xi)-\Lambda(\eta))} d\eta + c_1 \int_\lambda^x e^{-\rho\Lambda(\xi)} d\xi + c_2 \quad , \tag{332}$$

where

$$\Lambda(x) = \int_{\nu}^{x} b(x)\, dx \qquad (333)$$

and λ, μ, ν, c_1, c_2 are arbitrary constants, which are restricted by the prescribed boundary conditions. The five constants are, of course, not essential. In reality (332) depends only on two essential parameters, so that three of the five constants can be chosen arbitrarily. In order to obtain a form of the solution suitable for the boundary layer problem, it is convenient to set $\mu = \lambda$, while the choice of λ and ν shall be left undecided for the moment. For typographical reasons it is useful to introduce the abbreviations

$$P_s^t = \int_s^t d\xi \int_s^{\xi} \rho\, f(\eta) e^{-\rho(\Lambda(\xi)-\Lambda(\eta))} d\eta \qquad (334)$$

$$Q_s^t = \int_s^t e^{-\rho\Lambda(\xi)} d\xi \quad . \qquad (335)$$

Then (332) can be written

$$U(x,\rho) = P_\lambda^x + c_1 Q_\lambda^x + c_2 \quad , \qquad (336)$$

and substitution of the values $x = \alpha$ and $x = \beta$ into (336) leads to the two linear algebraic equations

$$0 = P_\lambda^\beta + c_1 Q_\lambda^\beta + c_2$$

$$0 = P_\lambda^\alpha + c_1 Q_\lambda^\alpha + c_2$$

for c_1 and c_2. Calculating c_2 from these equations and using the fact that $Q_\lambda^\beta - Q_\lambda^\alpha = Q_\alpha^\beta$, one finds that

$$c_2 = \frac{P_\lambda^\beta Q_\lambda^\alpha - P_\lambda^\alpha Q_\lambda^\beta}{Q_\alpha^\beta} \quad . \qquad (337)$$

investigation can be obtained immediately from (337) by the following considerations: Substitution of λ for x in (336) shows that $c_2 = u(\lambda)$. As λ was arbitrary, this is true for any value of λ, so that (337) can be regarded as the desired solution of (330) with λ instead of x as independent variable. Writing x for λ, the solution of the boundary value problem is therefore obtained in the form

$$U(x,\rho) = \frac{Q_x^\alpha}{Q_\alpha^\beta} P_x^\beta - \frac{Q_x^\beta}{Q_\alpha^\beta} P_x^\alpha \quad . \tag{338}$$

56. The asymptotic value of $\int_s^t F(x)e^{\rho\varphi(x)}dx$ for large ρ:

The solution of (336) is composed of integrals of the form

$$\int_s^t F(x)e^{\rho\varphi(x)}dx \quad . \tag{339}$$

It is therefore important to have asymptotic expressions for such integrals for large values of ρ. The following theorem, a proof of which can e.g. be found in a paper by O. Perron [8], will be the chief tool of the subsequent investigations.

Theorem: If $F(x)$ and $\varphi(x)$ are regular analytic in $s < x < t$, if

$$\varphi(x) \begin{cases} = 0, & \text{for } x = R,\ s < R < t \\ < 0, & \text{for } x \neq R,\ s < x < t \end{cases}$$

and if

$$\varphi(x) = (x-R)^p(g_0 + g_1(x-R) + \dots)$$

is the Taylor series of (x) around $x=R$, then

$$\int_s^t F(x)e^{\rho\varphi(x)}dx = [\frac{2}{p} F(R)\,\Gamma(\frac{1}{p})\,\frac{1}{|g_0|^{1/p}}]\,\rho^{-1/p} \quad , \tag{340}$$

where the brackets "[ ]" have the meaning defined in section 5. (The number p here has

where the brackets "[ ]" have the meaning defined in section 5. (The number p here has nothing to do with the number p of chapter I.) If $R = s$ or $R = t$, the same is true, but with the factor 2 in (340) missing.

This theorem can easily be generalized so as to include also the case $\varphi(x) > 0$:

<u>Theorem</u>: If $F(x)$ and $\varphi(x)$ are regular analytic in $s < x < t$, if $\varphi(R)$ is the maximum of $\varphi(x)$ in $s < x < t$, where $s < R < t$, and if

$$\varphi(x) = \varphi(R) + (x-R)^p(g_0 + (x-R)g_1 + \dots)$$

is the Taylor series of $\varphi(x)$ around $x = R$, then

$$\int_s^t F(x)e^{\rho\varphi(x)}dx = [\frac{2}{p} F(R)\Gamma(\frac{1}{p}) \frac{1}{|g_0|^{1/p}}]\rho^{-1/p} e^{\rho\varphi(R)} \quad . \tag{341}$$

If $R = s$ or $R = t$, the same is true with the factor 2 in (341) missing.

<u>Proof</u>: The integral

$$\int_s^t F(x)e^{\rho(\varphi(x)-\varphi(R))}dx$$

satisfies the assumptions of Perron's theorem with $\varphi(x) - \varphi(R)$ instead of $\varphi(x)$. Since

$$\varphi(x) - \varphi(R) = (x-R)^p(g_0 + (x-R)g_1 + \dots) \quad ,$$

(341) follows immediately, if (340), applied for the exponent $\varphi(x) - \varphi(R)$ is multiplied on both sides by $e^{\rho\,(R)}$.

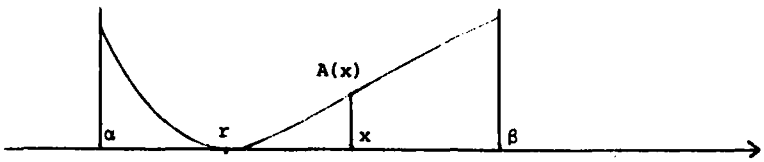
57. Passage to the limit in (338), if

$$b(x) \begin{cases} < 0 & \text{for} \quad \alpha < x < r \\ = 0 & \text{for} \quad x = r \\ > 0 & \text{for} \quad r < x < \beta \quad . \end{cases}$$

The asymptotic calculation of the integrals in (338) is easiest when the constant ν in the definition of $A(x)$, formula (333), is chosen equal to r. If this is done, the function $A(x)$ satisfies the following conditions:

$$\left.\begin{array}{l} A(x) > 0, \; x \neq r \\ A(x) = 0, \quad x = r \\ A(x) \text{ is monotonic increasing for } x > r \\ A(x) \text{ is monotonic decreasing for } x < r \end{array}\right\} \qquad (342)$$

$A(x)$ has therefore the shape indicated by the figure below.



The passage to the limit, as $\rho \to \infty$, in (338) leads to different results, according as x is less, equal or greater than r.

<u>Case a). $x > r$.</u>

Application of formula (341) to Q_α^x and Q_α^β show that for $x > r$ both integrals have the same asymptotic value, hence

$$\lim_{\rho\to\infty} \frac{Q_x^\alpha}{Q_\alpha^\beta} = - \lim_{\rho\to\infty} \frac{Q_\alpha^x}{Q_\alpha^\beta} = -1 \quad . \qquad (343)$$

To the inner integral of P_x^β formula (340) can be applied. For, in this case, $r < x \leq \eta \leq \xi$ and, in this range, $-(A(\xi) - A(\eta))$ as function of η assumes its maximum value 0 for $\eta = \xi$, on account of (342). As

$$\frac{\delta}{d\eta}(-A(\xi) + A(\eta) = -b(\eta)$$

is, by assumption, not zero, the number p of (340) is here equal to 1, hence

$$\int_x^{\xi} \rho\, f(\eta) e^{-\rho(A(\xi)-A(\eta))} d\eta = [f(\xi) \frac{1}{b(\xi)}]$$

and therefore

$$\lim_{\rho\to\infty} P_x^{\beta} = \int_x^{\beta} \frac{f(\xi)}{b(\xi)}\, d\xi \quad . \tag{344}$$

The expressions in brackets in (340) and (341) can be sure to be different from zero, if $F(x) \neq 0$ in $\alpha < x < \beta$. In order to avoid too lengthy formulas, this additional assumption shall temporarily be made. It is, however, by no means essential, and it will be shown later how to proceed without it.

The letter E in the subsequent formulas shall be used to indicate non-vanishing constants. Note that the same letter E will be used for different constants.

One finds immediately, by application of (341),

$$Q_x^{\beta} = [E]\, \frac{1}{\rho}\, e^{-\rho A(x)} \quad . \tag{345}$$

In

$$Q_\alpha^{\beta} = \int_\alpha^{\beta} e^{-\rho A(\xi)} d\xi$$

the exponent reaches its maximum 0 for $\xi = r$. As, by assumption, $b(r) = 0$, but $b'(r) \neq 0$, p is equal to 2 in this case and

$$Q_\alpha^{\beta} = [E]\, \frac{1}{\sqrt{\rho}} \quad . \tag{346}$$

In order to find the order of magnitude of P_x^{α}, consider that $\alpha < \xi < \eta < x$ and $r < x$ in the exponent $-(A(\xi) - A(\eta))$ occurring in P_x^{α}. Hence, the maximum of $-(A(\xi) - A(\eta))$ as function of η for fixed ξ is

$$-(A(\xi) - A(\xi)) = 0, \quad \text{for} \quad A(\xi) > A(x)$$

$$-(A(\xi) - A(x)) > 0, \quad \text{for} \quad A(\xi) < A(x) \quad .$$

The asymptotic value of the inner integral of P_x^{α} is

$$\int_x^{\xi} \rho\, f(\eta) e^{-\rho(\Lambda(\xi)-\Lambda(\eta))} d\eta = \begin{cases} [E] & , \text{ for } \Lambda(\xi) > \Lambda(x) \\ [E] e^{-\rho(\Lambda(\xi)-\Lambda(x))} & , \text{ for } \Lambda(\xi) < \Lambda(x) \end{cases} .$$

The contribution to P_x^{α} of that part of the interval $\alpha < \xi < x$ for which $\Lambda(\xi) > \Lambda(x)$, (if it exists), can be neglected in comparison with the part where $\Lambda(\xi) < \Lambda(x)$. As

$$\int_x^{\alpha} e^{-\rho(\Lambda(\xi)-\Lambda(x))} d\xi = [E] \frac{1}{\sqrt{\rho}} e^{\rho\Lambda(x)}$$

one has therefore

$$P_x^{\alpha} = [E] \frac{1}{\sqrt{\rho}} e^{\rho\Lambda(x)} \quad . \tag{347}$$

From (345), (346) and (347) it follows, finally, that

$$\frac{Q_x^{\beta}}{Q_{\alpha}^{\beta}} P_x^{\alpha} = [E] \frac{1}{\rho}$$

and therefore, using (344),

$$\lim_{\rho\to\infty} U(x,\rho) = -\int_x^{\beta} \frac{f(\xi)}{b(\xi)} d\xi, \quad \text{for} \quad x > r \quad . \tag{348}$$

<u>Case b). $x < r$.</u>

A consideration analogous to that used for case a) leads to

$$\lim_{\rho\to\infty} U(x,\rho) = \int_{\alpha}^{x} \frac{f(\xi)}{b(\xi)} d\xi, \quad \text{for} \quad x < r \quad . \tag{349}$$

See also case c) and d), section 58.

<u>Case c). $x = r$.</u>

If $\frac{f(x)}{b(x)}$ is regular analytic at $x = r$, then one shows immediately, by a similar consideration, that

$$\frac{1}{2} \{\int_{\alpha}^{r} \frac{f(\xi)}{b(\xi)} d\xi + \int_{\beta}^{r} \frac{f(\xi)}{b(\xi)} d\xi\}, \quad \text{for} \quad x = r \quad .$$

(The factor $\frac{1}{2}$ is due to the Q's in (338).) In other words, $U(x,\rho)$ tends in this case to the arithmetic mean of the two limits at $x = r$.

58. Passage to the limit in (338), if

$$b(x) \begin{cases} > 0, & \text{for} \quad \alpha < x < r \\ = 0, & \text{for} \quad x = r \\ < 0, & \text{for} \quad r < x < \beta \ . \end{cases}$$

In order to operate as much as possible with positive quantities it is convenient to set now

$$A(x) = \int_r^x b(x)\, dx \ . \tag{350}$$

Then $A(x)$ satisfies the conditions (342). If in the definition of P_s^t and Q_s^t the sign of the exponents is changed, $U(x,\rho)$ can again be written in the form (339).

In addition to the distinction between the cases $x > r$ and $x < r$, the relative size of $A(\alpha)$ and $A(\beta)$ plays now a part in the proof.

<u>Case a]. $x > r$, $A(\beta) > A(\alpha)$</u>.

Let $s > r$ be the value for x for which

$$A(s) = A(\alpha) \ .$$

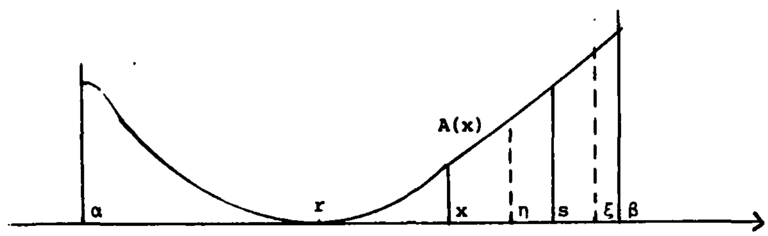
Then

$$Q_x^\alpha = \begin{cases} [E]\ \frac{1}{\rho}\, e^{\rho A(x)} & , \quad x > s \\ [E]\ \frac{1}{\rho}\, e^{\rho A(\alpha)} & , \quad x < s \end{cases} \tag{351}$$

and

$$Q_\alpha^\beta = [E]\ \frac{1}{\rho}\, e^{\rho A(\beta)} \ . \tag{352}$$

The relative positions of x, ξ and η in P_x^β are indicated in the figure below.



One sees that $\max(A(\xi) - A(\eta)) = A(\xi) - A(x)$ for fixed ξ. Therefore

$$\int_x^\xi \rho\, f(\eta) e^{\rho(A(\xi)-A(\eta))} d\eta = [E]\; e^{\rho(A(\xi)-A(x))}$$

$$\int_x^\beta e^{\rho(A(\xi)-A(x))} d\xi = [E]\; \frac{1}{\rho}\, e^{\rho(A(\beta)-A(x))}$$

and

$$P_x^\beta = [E]\; \frac{1}{\rho}\, e^{\rho(A(\beta)-A(x))} \quad . \tag{353}$$

Furthermore

$$\frac{Q_x^\beta}{Q_\alpha^\beta} = [E] \quad . \tag{354}$$

For P_x^α consider again the relative position of x, ξ and η. If $\xi > r$, the inner integral in P_x^α remains finite, as $\rho \to \infty$, hence only the case $\xi < r$ has to be considered. In that case

$$\max(A(\xi) - A(\eta)) = A(\xi) - A(r) = A(\xi)$$

and

$$\int_x^\xi \rho\, f(\eta)\; e^{\rho(A(\xi)-A(\eta))} d\eta = [E]\; \sqrt{\rho}\; e^{\rho A(\xi)} \quad .$$

As

$$\int_x^\alpha e^{\rho A(\xi)} d\xi = \begin{cases} [E]\; \dfrac{1}{\rho}\, e^{\rho A(x)} \quad , & x > s \\[2ex] [E]\; \dfrac{1}{\rho}\, e^{\rho A(\alpha)} \quad , & x < s \end{cases}$$

one has

$$P_x^{\alpha} = \begin{cases} [E] \dfrac{1}{\sqrt{\rho}} e^{\rho A(x)} & , \quad x > s \\ [E] \dfrac{1}{\sqrt{\rho}} e^{\rho A(\alpha)} & , \quad x < s \quad . \end{cases} \tag{355}$$

Substituting formulas (351) - (338) one obtains

$$U(x,\rho) = \begin{cases} [E] \dfrac{1}{\rho} & - [E] \dfrac{1}{\sqrt{\rho}} e^{\rho A(x)} \quad , \quad x > s \\ [E] \dfrac{1}{\rho} e^{\rho(A(\alpha)-A(x))} & - [E] \dfrac{1}{\sqrt{\rho}} e^{\rho A(\alpha)} \quad , \quad x < s \end{cases}$$

or

$$\lim_{\rho \to \infty} U(x,\rho) = \pm\infty \quad . \tag{356}$$

<u>Case b). $x > r$, $A(\beta) < A(\alpha)$</u> .

The reasoning is of the same type as in case a), only the orders of magnitude of the terms change. Let $s < r$ be the value of x for which

$$A(s) = A(\beta) \quad .$$

One finds

$$\frac{Q_x^{\alpha}}{Q_{\alpha}^{\beta}} = [E] \quad . \tag{357}$$

For P_x^{β} the asymptotic formula (353) holds unchanged. Furthermore

$$Q_x^{\beta} = [E] \frac{1}{\rho} e^{\rho A(\beta)} \tag{358}$$

$$Q_{\alpha}^{\beta} = [E] \frac{1}{\rho} e^{\rho A(\alpha)} \quad . \tag{359}$$

The relative positions of x, ξ and η in the inner integral of P_x^{α} can be seen in the figure below. Only the case $\xi < r$ has to be considered, as for $\xi > r$ the integral tends to zero.

AD-A103 857 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1

ON BOUNDARY LAYER PROBLEMS IN THE THEORY OF ORDINARY DIFFERENTI--ETC(U)

JUL 81 W R WASOW DAAG29-80-C-0041

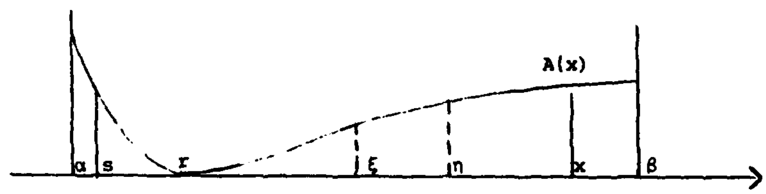
UNCLASSIFIED MRC-TSR-2244 NL

END

DATE FILMED

8

DTIC



For $\xi < r$

$$\int_x^{\xi} \rho\, f(\eta)\, e^{\rho(\Lambda(\xi)-\Lambda(\eta))} d\eta = [E]\ \sqrt{\rho}\, e^{\rho\Lambda(\xi)}$$

and

$$\int_x^{\alpha} e^{\rho\Lambda(\xi)} d\xi = [E]\ \frac{1}{\rho}\, e^{\rho\Lambda(\alpha)}$$

hence

$$P_x^{\alpha} = [E]\ \frac{1}{\sqrt{\rho}}\, e^{\rho\Lambda(\alpha)} \quad . \tag{360}$$

The expressions (353) and (357) - (360), inserted in (338) lead to

$$U(x,\alpha) = [E]\ \frac{1}{\rho}\, e^{\rho(\Lambda(\beta)-\Lambda(x))} - [E]\ \frac{1}{\sqrt{\rho}}\, e^{\rho\Lambda(\beta)} \quad .$$

Therefore

$$\lim_{\rho\to\infty} U(x,\rho) = \pm\infty \quad . \tag{361}$$

Case c) and d). $x < r$.

One might repeat the preceding arguments in analogous form and with the same final result for $x < r$. Instead, one can also proceed as follows: The transformation

$$z = r - x$$

changes the differential equation (330) into

$$\frac{1}{\rho}\frac{d^2y}{dz^2} + \tilde{b}(z)\,\frac{dy}{dz} = \tilde{f}(z) \tag{362}$$

where

$$\tilde{b}(z) = -b(r-z)$$
$$\tilde{f}(z) = f(r-z) \quad .$$

The boundary conditions $y(\alpha) = y(\beta) = 0$ are transformed into

$$y(r-\beta) = y(r-\alpha) = 0 \quad .$$

As

$$\tilde{b}(z) \begin{cases} > 0, & \text{for} \quad r-\beta < z < 0 \\ = 0, & \text{for} \quad z = 0 \\ < 0, & \text{for} \quad 0 < z < r-\alpha \quad . \end{cases}$$

The results of the preceding section can be applied to (362) and lead to

$$\lim_{\rho\to\infty} \tilde{U}(z,\rho) = \pm\infty \quad , \quad \text{for} \quad 0 < z < r-\alpha \quad .$$

But this is equivalent with

$$\lim_{\rho\to\infty} U(x,\rho) = \pm\infty \quad , \quad \text{for} \quad \alpha < x < r \quad .$$

This method might also have been used to prove formula (349).

59. Some minor generalizations:

1) If $f(x)$ is allowed to have roots in $\alpha < x < \beta$, the asymptotic values used for P_x^α and P_x^β are not correct for these values of x which are roots of $f(x)$. If, in particular, $f(r) = 0$, some of the expressions would even be incorrect for all x. The reason is that $f(x)$ or $f(r)$ appear in the constant factors occurring in these expressions. But in that case the calculation could be carried through with a slightly more general form of formula (340) also contained in Perron's general formula in the paper [8] quoted above. The result is again the same.

2) Non-homogeneous boundary conditions: If to (338) is added a solution of the homogeneous differential equation

$$\frac{1}{\rho} y'' + b(x)y' = 0 \tag{363}$$

satisfying the *non-homogeneous* boundary conditions

$$y(\alpha) = \ell_2, \quad y(\beta) = \ell_1 \tag{364}$$

one obtains the solution of (330) satisfying the boundary conditions (364).

As the transformation

$$y^* = y - \ell_2$$

changes (330) into a differential equation of the same type and transforms the non-homogeneous left hand boundary condition into a homogeneous one, it is no loss of generality to assume that

$$\ell_2 = 0 \quad .$$

The general solution of (363) can be written in the form

$$U(x,\rho) = c_1 \, \Omega_\alpha^x + c_2 \quad .$$

The given boundary conditions lead then to

$$c_1 = \frac{u_1}{\Omega_\alpha^\beta} \quad , \quad c_2 = 0 \quad ,$$

hence

$$U(x,\rho) = \frac{\Omega_\alpha^x}{\Omega_\alpha^\beta} \ell_1 \quad . \qquad (365)$$

To (365) the methods of sections 57 and 58 can be immediately applied with the following result:

a). $$b(x) \begin{cases} < 0, & x < r \\ > 0, & x > r \end{cases} ,$$

then

$$\lim_{\rho\to\infty} U(x,\rho) = \begin{cases} \ell_1 & \text{for } x > r \\ 0 & \text{for } x < r \end{cases}$$

b). $$b(x) \begin{cases} > 0, & \text{for } x < r \\ < 0, & \text{for } x > r \end{cases} ,$$

then

α.) $\Lambda(\beta) > \Lambda(\alpha)$,

$$U(x,\rho) = \begin{Bmatrix} [E] \; e^{\rho(\Lambda(x)-\Lambda(\beta))}, & x > s \\ [E] \; e^{\rho(\Lambda(\alpha)-\Lambda(\beta))}, & x < s \end{Bmatrix} + 0 \quad .$$

β.) $\Lambda(\beta) < \Lambda(\alpha)$,

$$\lim_{\rho\to\infty} U(x,\rho) = \ell_1 \ .$$

60. <u>Theorem 7</u>:

Given the differential equation

$$\frac{1}{\rho} y'' + b(x)y' = f(x) \tag{330}$$

where $b(x)$ and $f(x)$ are regular analytic functions in the interval $\alpha < x < \beta$ and $b(x)$ has exactly one root $x = r$ in the interior of the interval, while $b'(r) \neq 0$. Then the behavior for great values of ρ of the solution of (330) which satisfies the boundary conditions

$$y(\alpha) = \ell_2, \ y(\beta) = \ell_1 \tag{331}$$

depends essentially on the shape of $b(x)$:

1). If $b(x) \begin{cases} < 0 & \text{for} \quad \alpha < x < r \\ > 0 & \text{for} \quad r < x < \beta \ , \end{cases}$

then the solution $U(x,\rho)$ converges with increasing ρ in the whole interval $\alpha < x < \beta$, except possibly at $x = r$, the limiting function being composed of the two solutions of the limiting differential equation of the first order

$$b(x)y' = f(x)$$

satisfying one of the two prescribed boundary conditions. If these functions are bounded at $x = r$, then the solution of (330) converges at $x = r$ to the arithmetic mean of the two limits at this point.

2). If $b(x) \begin{cases} > 0 & \text{for} \quad \alpha < x < r \\ < 0 & \text{for} \quad r < x < \beta \ , \end{cases}$

then $U(x,\rho)$ diverges with increasing ρ at all points of the interval.

Appendix

A SHORT REPORT ON THE ASYMPTOTIC SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS INVOLVING A PARAMETER

61. The main mathematical tool used in this paper is the theory of the asymptotic solution of differential equatins involving a parameter ρ for large values of this parameter. The most important results of this theory are contained in papers by G. D. Birkhoff [1], Noaillon [2], Tamarkin [3], [4], Langer [5], and Trjitzinsky [6]. The asymptotic developments used by Birkhoff and Tamarkin, *although of a very general character,* do not apply to the particular differential equations of this investigation, because they assume that a certain "characteristic" algebraic equation formed with the coefficients of the differential equation has no multiple roots, an assumption not satisfied in our case.

It would probably not be difficult to modify the methods used by Birkhoff and Tamarkin in such a way that they cover our case. But this is not necessary, since the type of differential equations considered by Noaillon and Trjitzinsky includes the differential equation (101).

A complete proof of the main theorem of Noaillon and Trjitzinsky would be beyond the scope of this investigation, even if we restricted ourselves to the special case in which we are interested. We intend here only to give a summarized report on the methods of this theory and to show how, assuming the theorems proved, the asymptotic expressions of theorem 1 can be obtained in our case.

The theory consists of two parts. In the first part, which may be called the "formal part", the "exact" differential equation

$$L(y,\rho) = 0 \qquad (401)$$

which is essentially equivalent to the differential equation

$$\frac{L(y,\rho)}{y} = 0 \qquad (402)$$

is replaced by the "asymptotic" differential equation

$$\frac{L(y,\rho)}{y} \underset{j}{\sim} 0 \ . \qquad (403)$$

Here the symbol "$\underset{j}{\sim}$" has the following meaning:

$$f(x,\rho) \underset{j}{\sim} g(x,\rho) \quad ,$$

where j is a real number, stands for

$$f(x,\rho) - g(x,\rho) = \frac{E(x,\rho)}{\rho^{-j}} \quad .$$

As previously, $E(x,\rho)$ is a function such that there is a positive real number R so that $|E(x,\rho)|$ is uniformly bounded for $\alpha \leq x \leq \beta$ and $|\rho| > R$. Usually, $f(x,\rho)$ will be regarded as asymtotically equal to $g(x,\rho)$ only if j is positive. But sometimes, e.g. in the case of the differential equation (101), a solution of (403) with a negative j will be an asymtotic approximation in the ordinary sense of the exact diferential equation (401).

It is then shown that under very general assumptions a function $Y(x,\rho)$ can be constructed which satisfies the condition (403), provided the number j is not too large.

If certain differentiability conditions are satisfied, j may have an arbitrarily large value. This is the case treated in detail by Noaillon and Trjitzinsky. The case in which there is an upper limit for j is only mentioned occasionally by these authors. But since we are only interested in the first terms of the resulting asymptotic expansions, it is unnecessary to assume indefinite differentiability of the coefficients of the differential equation. This assumption is required only, if we are interested in the unlimited asymptotic expansion. Going over Noaillon's proof it is easily seen that assumption 4° of Noaillon's theorem in section 62 below is sufficient to guarantee the existence of the first term of the asymptotic solution.

In the second, the "functional" part it is shown that the solutions of the asymptotic differential equation (403) are asymptotically equal to the solutions of the exact differential equation (401).

Essential for our application is furthermore the result that the derivatives of these asymptotic solutions of (401) are asymptotically equal to the derivatives of the corresponding exact solutions.

The complete statement of the results of Noaillon's paper, as far as they are important for our purpose, follows:

62. Noaillon's Theorem.

Part I: Given is the differential equation

$$L(y,\rho) = \sum_{i=0}^{n} P_i(x,\rho)y^{(n-i)} = 0 \tag{404}$$

satisfying the following conditions:

1°. ρ is a real positive parameter.

2°. x is a real variable

3°. In the domain $\alpha < x < \beta$, $\rho > R$ (α, β, R constants) the coefficients $P_i(x,\rho)$ can be expanded in convergent series of the form

$$P_i(x,\rho) = \rho^{H_i} \sum_{s=0}^{\infty} B_{is}(x)\rho^{-s} \ . \tag{405}$$

(The H_i are positive integers.)

4°. The functions B_{is} have at least n continuous derivatives in $\alpha < x < \beta$.

5°. The coefficients $B_{00}(x)$ in (405) does not vanish in any point of the interval $\alpha < x < \beta$.

To these conditions 1° - 5° a sixth assumption has to be added, which can be most easily defined in the course of the construction of the asymptotic solution.

If these conditions are satisfied, then there can be constructed solutions $Y(x,\rho)$ of (403), each of which can be written in the form

$$Y(x,\rho) = T\cdot u \tag{406}$$

where the "principal factor" T is a function of the form

$$T = e^{\int_{x_0}^{x} \psi(\xi,\rho)d\xi} \tag{407}$$

with

$$\psi(x,\rho) = \sum_{i=1}^{p} \lambda_i(x)\rho^{\alpha_i} \tag{408}$$

the α_i being non-negative decreasing rational numbers and p being a positive integer independent of j. u stands for the "secondary factor"

$$u(x,\rho) = \sum_{\nu=0}^{j'} E_\nu(x)\rho^{-\frac{\nu}{M}} . \tag{409}$$

Here M is a positive integer independent of j. The positive integer j' depends on j and increases with j. (In the application to differential equation (101) the first term of (409) is obtained if $j = -\frac{m-2}{n-m}$, as we shall see.)

As we have said before, the maximal value of j for which a solution $Y(x,\rho)$ of (403) can be constructed depends on the number of times the coefficients B_{is} can be differentiated. It can be only determined in the course of the successive construction of the terms of (409). If the B_{is} can be differentiated indefinitely, then j and j' can have arbitrarily large values.

Remark: The theory remains valid if the series (405) are not convergent but only asymptotic expansions. But we do not need this case for our application.

<u>Functional Part</u>: Let $Y(x,\rho) = T\cdot u$ be a solution of (403). Then there is a solution $y(x,\rho)$ of the exact differential equation (401) such that

$$y(x,\rho) = T(u + \rho^{-\frac{j'}{M}}[0]) \tag{410}$$

and this equation can be formally differentiated at least $n-1$ times, i.e. it can be differentiated treating the symbol $[0]$ as if it were a constant.

63. The construction of the asymptotic solution of the differential equation:

A) <u>The principal factor</u>.

The first step: In $\frac{L(y)}{y}$ substitute an expression of the form

$$Y = T\cdot u$$

(compare (407), (408), (409)). The result of this substitution is an expression of the form

$$\sum_{i=1}^{k} f_i(x)\rho^{\gamma_i} + \rho^{\gamma_k}[0] \tag{411}$$

the first term $A_1(x)\rho^{\alpha_1}$ in (408) in such a way that the coefficients $f_1(x)$ of the term of highest order in (411) vanishes.

Noaillon gives a general method that allows us to determine the exponent α_1 and the function $A_1(x)$ systematically in the general case. But our application being of such a simple type, α_1 and $A_1(x)$ can be found, in our case, more directly.

Second step: Having chosen the first term $A_1\rho^{\alpha_1}$ of ψ we set

$$\rho = \rho_1^{q_1}$$

$$y = e^{\int_{x_0}^{x} A_1(\xi)\rho^{\alpha_1} d\xi}\, y_2$$

where q_1 is the denominator of the rational number α_1. This substitution transforms the expression $\frac{L(y)}{y}$ into an expression

$$\frac{L_2(y_2)}{y_2}$$

We now repeat the reasoning of the first step with $\frac{L_2(y_2)}{y_2}$ instead of $\frac{L(y)}{y}$, in order to find the term $A_2\rho^{\alpha_2}$, considering, of course, for α_2 only values that are smaller than α_1. We continue in this manner until we arrive at the last term $A_p\rho^{\alpha_p}$ for which α_p is still positive. That there is such a term, i.e. that we always attain an exponent $\alpha_{p+1} \leq 0$ after a finite number of steps, is proved in Noaillon's paper.

B) The secondary factor.

First step: In $L(Y)$ substitute $Y = T \cdot u$ where T is the function calculated in A) and u as yet undetermined. The result of the substitution is an expression of the form $T\,K(u,\rho)$, where $K(u,\rho)$ is a linear differential expression in u whose coefficients are power series in $\sigma = \rho^{1/M}$, M being the common denominator of all the exponents α_i in ψ. In $K(u,\rho)$ collect the terms of highest order in σ. Then $K(u,\rho)$ can be written in the form

$$K(u,\rho) = S\,G(u) + H(u,\rho) \quad . \qquad (412)$$

Here S is the highest power of σ occurring in $K(u,\rho)$ and $G(u)$ and $H(u,\rho)$ are differential expressions, $G(u)$ being independent of ρ. It can be proved that the highest order of differentiation occurring in $G(u)$ is greater than zero.

Second step: Find a solution of the differential equation

$$G(u) = 0 \quad . \qquad (413)$$

We take this solution as the first term $E_0(x)$ of the series (409). Since we want $E_0(x)$ to be bounded in the whole interval of x in which we consider the asymptotic expansion, we have to add to the assumptions 1° - 5° in section 62 the condition 6°: The coefficient of the highest derivative in (413) does not vanish in any part of the interval $\alpha \leq x \leq \beta$.

Third step: In order to find the function $E_\mu(x)$, $\mu > 0$ of (409), Noaillon proceeds as follows: He determines by recursion a sequence of functions $w_1(x,)$ $w_2(x,),\ldots$ from the formula

$$G(w_\nu) = \frac{1}{S} H(w_{\nu-1},\rho) \quad . \qquad (414)$$

It is easily seen that the $w_\nu(x,\rho)$ are of the form

$$w_\nu = \sum_{i=0}^{\infty} S^{-i} g_{\nu i}(x) \quad . \qquad (415)$$

The functions $E_\mu(x)$ are then given by

$$E_\mu(x) = \sum_{\nu=1}^{\mu} g_{\nu\mu}(x) \quad . \qquad (416)$$

It is not difficult to prove that the function (406), if determined by the construction which we have just outlined here, does in fact satisfy the relation (403).

The construction of the asymptotic solution $Y(x,\rho)$ is by no means uniquely determined, and it can be proved that the construction yields asymptotic expansions which are asymptotic approximations to a fundamental system of solutions of (401).

64. Asymptotic Solution of $L(y) \equiv \frac{1}{\rho} N(y) + M(y) = 0$.

A) The principal factor.

First step: We now apply the construction described in section 63 to the particular type of differential equation under consideration. Substituting

$$y = Y = T \cdot u$$

in $\frac{L(y,\rho)}{y}$, where $L(y,\rho)$ is now the differential expression (101), and T and u are expressions of the form described by (407), (408), (409), we see that, unless $\psi = 0$, the term of highest order of $\frac{y^{(s)}}{y}$ is $\left(A_1(x)\rho^{\alpha_1}\right)^s$. The condition that the highest terms of $\frac{1}{\rho}\frac{N(y)}{y}$ as function of ρ, and of $\frac{M(y)}{y}$ cancel out is therefore

$$A_1^n(x)\rho^{\alpha_1 n-1} \equiv -b_0(x)\, A_1^m(x)\rho^{\alpha_1 m} ,$$

or

$$\alpha_1 n - 1 = \alpha_1 m \tag{417}$$

and

$$A_1^n = -b_0 A_1^m , \tag{418}$$

provided $\psi \neq 0$. From (417) and (418) we conclude

$$\alpha_1 = \frac{1}{n-m} \tag{419}$$

and

$$A_1 = (-b_0)^{1/n-m} . \tag{420}$$

Second step: Now we write

$$y = e^{\sigma \int_{x_0}^{x} (\xi)d\xi} \, Y_2 \tag{421}$$

where σ is defined by $\rho = \sigma^{n-m}$ and (x) is one of the functions $_\nu(x)$ defined in theorem 1, section 6. Y_2 is the function

$$Y_2 = e^{\int_{x_0}^{x} \psi_2(\xi)d\xi} \cdot u . \tag{422}$$

Here u is again the series (409) and

$$\psi_2(x) = \sum_{i=2}^{p} A_i(x)\rho^{\alpha_i} . \tag{423}$$

We find

$$\frac{y^{(\nu)}}{y} = \sigma^{\nu}\,{}^{\nu}(x) + \nu\,\frac{\sigma^{\nu-1}\varphi^{\nu-1}(x)\,y_2'}{y_2} + \cdot\cdot\cdot\cdot\cdot \quad . \tag{424}$$

The term of highest order in the second term of the right member, if Y_2' is replaced by its value according to (422) and (423) has the value

$$\nu\, A_2\, \sigma^{\nu-1}\, \rho^{\alpha_2}\, \varphi^{\nu-1} \quad .$$

This term is of lower order in ρ than the term $\sigma^{\nu}\varphi^{\nu}$, since

$$\rho^{\alpha_2}\,\sigma^{\nu-1} = \sigma^{\nu+(n-m)\alpha_2-1}$$

and

$$(n-m)\alpha_2 - 1 < 0 \quad ,$$

in consequence of the assumption

$$\alpha_2 < \alpha_1 = \frac{1}{n-m} \quad .$$

The omitted terms in (424), which are indicated by dots, are similarly seen to be of still lower order. The terms of highest order in $\frac{L(y_2)}{y_2}$, i.e. $\frac{1}{\rho}\sigma^n\varphi^n$ and $b_0\sigma^m\varphi^m$ cancel out, since $\varphi(x)$ and σ have been chosen such as to achieve just this. The next terms are

$$n\,\frac{1}{\rho}\,A_2\sigma^{n-1}\rho^{\alpha_2}\varphi^{n-1} \quad , \quad \text{for} \quad \frac{1}{\rho}\,\frac{N(y_2)}{y_2} \tag{425}$$

and

$$m\, b_0\, A_2\sigma^{m-1}\rho^{\alpha_2}\varphi^{m-1} \quad , \quad \text{for} \quad \frac{M(y_2)}{y_2} \quad , \tag{426}$$

provided $\alpha_2 > 0$. (For, if $\alpha_2 = 0$, there are more terms of the same order as (425) and (426).)

Following Noaillon's construction we try to determine α_2 and A_2 in such a way that these two terms cancel out. But setting the sum of (425) and (426) equal to zero, and inserting for $\varphi(x)$ its value $(-b_0(x))^{1/n-m}$ leads to

$$n-m = 0 \quad ,$$

which was excluded.

Hence, $\alpha_2 > 0$ is impossible and therefore $\psi = \sigma\varphi(x)$.

B) The secondary term.

We follow the construction of Noaillon in order to find the differential expression $K(u,\rho)$ of (412). Since

$$y^{(\nu)} = T(\sigma^{\nu}\varphi^{\nu}u + \nu\sigma^{\nu-1}\varphi^{\nu-1}u' + \cdot\cdot\cdot\cdot)$$

where the dots indicate terms of lower order, we find by an easy calculation

$$L(Tu) = T\,K(u,\rho) = T\{\sigma^{m-1}(a_1\varphi^{n-1}u + n\varphi^{n-1}u' + b_1\varphi^{m-1}u + mb_0\varphi^{m-1}u') + H(u,\rho)\} \; .$$

Hence, the function $E_0(x)$ in this case is a solution of the differential equation obtained by setting the factor of σ^{m-1} equal to zero. This differential equation can be written

$$(a_1\varphi^{n-m} + b_1)u = (n\varphi^{n-m} + mb_0)u'$$

or

$$(a_1\,b_0 - b_1)u = b_0(n - m)u' \; .$$

Therefore

$$E_0(x) = \eta(x) = e^{-\int_{x_0}^{x} \frac{a_1(\xi)b_0(\xi) - b_1(\xi)}{b_0(\xi)(n-m)} d\xi} \; . \tag{427}$$

We are not interested in the other terms $E_\nu(x)$, $\nu > 0$ of (409).

Conditions 1° - 6° of Noaillon's theory are satisfied in our application for the whole interval $\alpha < x < \beta$. Condition 5°, in particular, is equivalent to condition 6° of the Main Theorem of chapter I. Hence, we conclude that there are n-m solutions of (101) of the form

$$Y_\nu(x,\rho) = e^{\sigma\int_\alpha^x \varphi_\nu(\xi)d\xi}[\eta(x)],\cdot(\nu = 1,2,\ldots,n-m) \; . \tag{428}$$

Note that the function $\eta(x)$ is the same for all $Y_\nu(x, \;)$ and that it does not vanish in $\alpha < x < \beta$.

But we can find more asymptotic solutions of the differential equation (101) by dropping the assumption that the $\psi(x,\rho)$ of (408) is not zero. In fact, if the principal factor T of (407) is equal to 1, the method used for the construction of the secondary factor in section 63, B) leads to asymptotic solutions given by series of the form (409). The first term of each of these series is a solution of the differential equation $M(y) = 0$. Taking a fundamental system $u_\nu(x)$ of m independent solutions of this differential equation as first terms of m asymptotic solutions of (101) we can add to the $n-m$ solutions of (101) given by (428) m more solutions of the form

$$Y_{n-m+\nu}(x,\rho) = [u_\nu(x)] \quad , \quad (\nu = 1,2,\ldots,m) \quad . \tag{429}$$

linerly independent, for sufficiently large ρ, can be given by calculating the asymptotic value of the Wronskian of these n functions. It does not offer any difficulties.

This finishes the proof of theorem 1 of section 6.

REFERENCES

[1] G. D. Birkhoff, On the asymptotic character of the solution of certain linear differential equations containing a parameter. Tr. Am. Soc. 9, 1908, pp. 219-231.

[2] P. Noaillon, Dévéloppements asymptotiques dans les equations linéaires à paramètre variable. Mém. de la Soc. Roy. des Sc. de Liège, Troisiòme série, tome IX (1912), 197 pages.

[3] J. D. Tamarkin, On some general problems of the theory of ordinary linear differential equation..... (in Russian). Petrograd, 1917.

[4] J. D. Tamarkin, Some general problems of the theory of ordinary linear differential equations and expansions of an arbitrary function in series of fundamental functions. Math. Zeitschrift, vol. 27 (1927) pp. 1-54.

[5] P. E. Langer and G. D. Birkhoff, The boundary problems and developments..... . Proc. Am. Acad. Arts and Sciences, vol. 58 (1923), pp. 51-128.

[6] W. J. Trjitzinsky, The theory of linear differential equations containing a parameter, Acta Math. 67, p. 1-50.

[7] E. Rothe, Asymptotic solution of a boundary value problem. Iowa State College Journal of Sc., vol. XIII, no. 4 (1939).

[8] O. Perron, Ueber die naeherungsweise Berchnung von Funktionen grosser Zahlen. Sitzungsberichte der Bayer. Akad. der Wissenschafton, math.-phys. Klasse, 1917, pp. 191-212.

[9] K. O. Friedrichs and J. J. Stoker, The non-linear boundary value problem of the buckled plate, Am. J. of Math, v. 63 (1941).

WRW/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2244	2. GOVT ACCESSION NO. AD-A103 857	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On Boundary Layer Problems in the Theory of Ordinary Differential Equations		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Wolfgang R. Wasow		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE July 1981
		13. NUMBER OF PAGES 107
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Ordinary differential equations, Parameter, Boundary conditions, Asymptotic approximations, Boundary layers.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This research concerns linear ordinary differential equations depending in such a way on a parameter ρ that the "limit" differential equation obtained by letting ρ tend to ∞ in the differential equation is of lower order than the original one.

Adopting a term customary in physics we used the name boundary layer problem for the question: What happens to the solution of a boundary value problem of such a differential equation, if the parameter tends to ∞ in this solution?

(Abstract continued on next page)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (continued)

We gave a general answer to this question for the differential equation $\frac{1}{\rho} N(y) + M(y) = 0$, where $N(y)$ and $M(y)$ are linear differential expressions of order n and m, respectively $(n > m)$, and for non-homogeneous boundary conditions which consist in prescribing the values of derivatives (but not of linear combinations of such derivatives) at the endpoints. The question whether the solution of such a boundary value problem converges to a solution of the limiting differential equations, as $\rho \to \infty$, and what boundary conditions are satisfied by the limit function could be decided by an easily applicable rule. This rule showed, among other things that the solution converges only, if the prescribed n boundary conditions are not too unevenly distributed between the two endpoints.

If the order m of the limiting differential equation is only one less than the order n of the original differential equation, then the above mentioned rule could be extended to more general types of boundary conditions and also to non-homogeneous differential equations.

Since the most important boundary layer problems in the applications are concerned with systems of differential equations, we gave a simple example for the mathematical treatment of a boundary layer problem for a linear system of two simultaneous differential equations.

The validity of the general rule proved in this research was seen to be restricted by the assumption that the coefficient of the term of highest order of differentiation in $M(y)$ has no zeros in the interval of integration. In a special example we showed that interesting results can be obtained, if this assumption is dropped.

The theory of the asymptotic expansion of the solutions of linear differential equations involving a parameter, developed by G. D. Birkhoff, Noaillon, Tamarkin, Trjitzinsky and others proved an important and powerful tool in these investigations.